AN INTRODUCTION TO MODERN MATHEMATICS

Miss USHRI GUHA

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Preface

This text is an attempt on the part of the author to present a unified, sequential, albeit elementary development of what is popularly known as "Modern Mathematics". It endeavours at an attainment of "mathematical maturity" for the freshers in the colleges or the seniors in the schools, depending on the course the students take, while taking care of developing their skill at computation.

It is an elementary exposition of the topics discussed and acquaints the students with general ideas of mathematical structures and proofs of these topics with a fair amount, although not too much of mathematical rigor. A number of exercises, distributed at strategic places are aimed at helping the students delve deeper into the ideas presented in the articles.

Although the book is mainly intended for the Higher Secondary Students of Vidarbha Board and the Pre-University students of Nagpur University it can be successfully used in any elementary introductory course of Set theory, Functions, Binary operations etc.. The articles and problems which are marked with asterisks are outside the scope of the syllabus of the above mentioned H. S. C. and Pre-University courses, but each of them illustrates some fundamentally important concept.

The author wishes to take this opportunity to express her deepest gratitude to Dr. B. S. Fadnis, Professor and Head of the Department of Mathematics of Nagpur University without whose initiative, encouragement, guidance and overall help this book would neither have been started nor ever finished.

The author is also grateful to Prof. P. W. Kane, the Head of the Department of Mathematics, Hislop College, for going through a part of the manuscript and making valuable suggestions.

Thanks of the author are due also to her student Pradip Summanwar for copying a part of the manuscript and her sister Ushoshi Guha Thakurta for checking answers to most of the problems.

Any suggestion, opinion, criticism, advice about the book either by students or by teachers, will be most welcome.

Nagpur
3rd March, 1971

U. G.

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CHAPTER ONE

Sets, Algebra of Sets, Venn's Diagrams

1·1 Introduction :

George Cantor (1845-1918) in the beginning of this century developed an idea which revolutionised the world of Mathematics, resetting, beautifying and clarifying the pattern of nearly every branch of the mathematical structure. This idea–that of sets–opened up new avenues in mathematical thought and discipline and accelerated the progress of Mathematics at a tremendous speed. In fact, the set theory is at present being used not only in every branch of Mathematics, but in other allied subjects too.

1·2 Idea of a set :

Consider the following list :

England, India, France, Belgium.

This is a collection of four nations. Consider similarly these collections.

(i) Delhi, Bombay, Calcutta, Madras.
(ii) Rose, Lily, Lotus, Chrysanthemum.
(iii) Gopal, Madhu, Ramesh.

(iv) All integers between and excluding 2 and 10.
(v) The vowels of the English alphabet.

Each of these is a collection. (i), (ii) and (iii) are expressed by actually listing the objects and (iv) and (v) by fixing the property by which we can make the list if we so desire. Either way, each is a collection of certain objects. Moreover, there is no ambiguity in determining the objects of the collection in any of these examples. Such a collection in Mathematics is called a "set". Thus, a *set* is a well defined collection or class of distinct objects.

1·3 Well definedness and distinctness are the two essential characteristics that must be present in a collection if it is to be a set. A collection is said to be well defined provided it is so expressed that having given any particular object, we are able to decide whether or not it belongs to the collection. This criterion satisfied, a set may contain any type of objects as its elements. The properties of, and the laws obeyed by a set do not depend on the nature of the elements it contains. The following are some examples of collections which are sets

(i) The collection of all natural numbers.
(ii) The languages spoken in India.
(iii) The books on a particular shelf.
(iv) The group of students in a certain class room.

In the above examples, (i) and (ii) are collections in which the elements are objects of thought, whereas (iii) and (iv) are collections in which the elements are objects of perception. However, since each of these collections is well defined, each is therefore a set

Now consider the statement " The capital cities of ". We cannot enlist the objects satisfying this statement unless we know of what country the capital cities should be. Thus, we cannot form a set out of it and hence the collection that this statement may point at, is not a set. Consider similarly the expression " The ten most noble hearted men in the world. " Though as far as the English grammar is concerned the sentence is complete, it does not help us decide unambiguously as to which men should belong to this collection; for we do not know how to decide the " most noble hearted men. " Hence this statement, though forming a collection, dose not form a well defined collection, and hence does not form a set.

If a collection is to be a set then besides being well defined, the elements of this collection must be distinct. That is, one object will not be written more than once in the list. For example, the set of integers satisfying the equation $[x-2][x^2-5x+6]=0$ is the set of integers 2, 3, 2, but we shall always write it as the set of integers 2, 3

1·4 Notation :

We adopt the convention of using capital letters A, B, C etc. to denote sets, and small letters a, b, x, y etc to denote the objects or elements of the sets.

If the object x belongs to a set A, we call x a member of A and write this as $x \epsilon A$. The symbol 'ϵ' stands for the words " belongs to " or " is a member of ." Given a set A and any element x, two possibilities arise. Either [i] $x \epsilon A$ or [ii] $x \notin A$ This rule is called Law of Dichotomy.

1·5 There are two ways of expressing a set

1. Roster form or Tabular form :

When the actual elements of the set are given, we put them within braces and call it a set; for example, the set A of integers 1, 2, 3, 4 is written as $A=\{1, 2, 3, 4\}$

2. Defining property or set builder form :

When the actual elements forming the set are not given, but a rule is given with the help of which the elements of the set can be found, it is said to be in a set builder form. For example, " A is the set of all those positive integers which are less than 5", gives us the same set as in [1] of this article, only expressed in a different way. This is written as

$A=\{x/x$ is a positive integer and $x<5\}$. The line standing for the words " such that. '

1·6 The forms mentioned above are sometimes convertible to each other. For example,

(i) $\{x / x^2 - 3x + 2 = 0\}$ can be written as $\{2, 1\}$

(ii) $\{x / x + 5 = 5\}$ can be written as $\{0\}$

(iii) $\{y / y$ is an integer and $1 < y \leqslant 7\}$ can be written as $\{2, 3, 4, 5, 6, 7\}$

(iv) $\{m / m$ is a vowel in the word "recapitulation" $\}$ can be written as $\{a, e, i, o, u\}$

In a set, the order in which the elements are written within the braces is immaterial. Thus, $\{1, 9, 7, 5\}$, $\{1, 5, 7, 9\}$, $\{7, 5, 9, 1\}$ all denote the same set.

1·7 Equality of sets (Principle or axiom of extension).

When two sets contain precisely the same elements, they are said to be equal. Thus, if every element of a set A is also present in another set and conversely, then the set A is said to be equal to the set B and we write A = B.

e. g. (i) The sets $\{7, 5, 7, 6\}$, $\{5, 6, 5, 7\}$, $\{5, 6, 7\}$ are equal to one another.

(ii) Let $A = \{x / x^2 - 3x = -2\}$, $B = \{2, 1\}$,

$G = \{1, 2, 2, 1\}$ Then $A = B = G$.

(iii) Let $D = \{m / m^2 - 8\text{m} + 16 = 0\}$, $E = \{y / y - 4 = 0\}$

$F = \{4\}$ then $D = E = F$

(iv) Let $A = \{x / x$ is a letter in the word "follow" $\}$

$B = \{y / y$ is a letter in the word "flow" $\}$

C = The set of letters in "wolf"

$D = \{0, l, f, w\}$

Then $A = B = C = D$.

1·8 Singleton or unit set :

A set containing only one element is called a unit set or a singleton set.

e. g. (i) The set of integers satisfying the equation $x - 5 = 0$ i.e. $\{ 5 \}$ is a unit set.

(ii) The set of days of the week beginning with the letter *w* is { wednesday } and is a singleton set.

(iii) The set of points in a plane in which two straight lines intersect is a set containing only one point and is therefore, a singleton set.

1·9 Null set or void set :

Consider the set $\{ x / x^2 + 4 = 0$ and x is an integer $\}$. How many elements does this set have ? Obviously none. In the tabular form this set is therefore written as $\{ \ \}$ It is called a void set or null set or empty set and is denoted by the symbol ϕ (pronounced as ' phi. ') Some other examples of a null set are :

(i) The set of green cows.

(ii) The set of four footed men.

(iii) The set of integers between 7 and 8.

It should be clearly understood that a null set is different from $\{ 0 \}$, the latter being a set containing the only element 0.

It must also be noted that 2 and $\{ 2 \}$ are not the same. 2 is merely a number (or a symbol). Left alone, there is no idea at this stage, of a set connected with the symbol 2 whereas $\{ 2 \}$ means a set whose only element is the number 2.

1·10 Disjoint sets :

If two sets A and B are such that none of the elements of A occur as an element of B and conversely, then they are said to be disjoint sets.

e. g. (i) Let $X = \{ 2, 9, 0 \}$, $Y = \{ 5, 1, 3 \}$. Then X and Y are disjoint sets.

(ii) Let A be the set of boys in St. John's School, and B be the set of boys in St. Francis De Sales School. Then A and B are disjoint sets, for no boy of one school can be a student of the other school too.

(iii) Let E be the set of boys below 8 years in age, and F be the set of boys who are 12 years old. Then E and F are disjoint sets.

1·11 Finite and infinite sets :

A set is said to be "finite" if it contains either none or a specific number of elements; that is, if in counting the different members of the set the process of counting comes to an end. Otherwise a set is said to be infinite. Consider, for example, the following sets.

(i) The set of months in a year

(ii) $\{1, 5, 9, 3\}$

(iii) The set of all integers.

(iv) The lines drawn parallel to a given line in a plane.

(v) The set of all points lying on the circumference of a circle.

(i) and (ii) are finite sets, whereas (iii), (iv) and (v) are infinite sets.

1·12 Set inclusion (Subset and superset)

If every element of A is in B, then A is said to be subset of B and B is said to be superset of A.

Symbollically, this is written as $A \subseteq B$ or $B \supseteq A$.

e. g. (i) Let $A = \{2, 3, 4, 5\}$; $B = \{2, 3\}$; $C = \{2\}$.

Then, $B \subseteq A$, $C \subseteq B$, $C \subseteq A$.

(ii) Let A be the set of all right angled triangles and B be the set of all triangles.
Then $A \subseteq B$.

(iii) Let M be the set of all letters in English alphabet
K be the set of all vowels in English alphabet.
R be the set of all consonants in English alphabet.

Then $K \subseteq M$ and $R \subseteq M$.

We have seen that for A to be a subset of B, all that is needed is, every element of A should be in B. Every element of B may or may not occur in the set A. If every element of B is also in A, then obviously $B \subseteq A$. And then we have every element of A is in B and conversely, so that the definition of equality is satisfied and hence $A = B$.

Thus $A \subseteq B$ and $B \subseteq A$ is equivalent to $A = B$.

1·13 Subset and proper subset :

If every element of A is in B but every element of B is not in A, then $A \subseteq B$ but $B \nsubseteq A$ (i.e. B is not a subset of A) so that $A \neq B$. In such a case A is said to be a proper subset of B and we denote it as $A \subset B$, showing that the possibility of A being equal to B does not exist.

For example, (i) Let $A = \{5, 8, 1\}$ and $B = \{5, 9, 6, 0, 8, 4, 1\}$

Then $A \subset B$

(ii) Let N be the set of all positive integers.
I be the set of all integers.
R be the set of all real numbers.

Then $N \subset I$ and $I \subset R$

1·14 For any set A, $\phi \subseteq A$

Proof : Let, if possible, $\phi \nsubseteq A$. $\therefore$ There must be at least one element in ϕ which is not in A. But there cannot be any element in ϕ. Therefore $\phi \nsubseteq A$ is false. $\therefore \phi \subseteq A$.

1·15 The subset relationship obeys the following laws.

1. Reflexivity : $A \subseteq A$ for any set A.

Since any element of A is in A itself whatever be the set A, $\therefore A \subseteq A$.

2. Antisymmetry : If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

$\because A \subseteq B \therefore$ every element of A is in B.

$\because B \subseteq A \therefore$ every element of B is in A.

$\therefore$ By definition of set equality, $A = B$.

3. Transitivity : If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

$\because A \subseteq B \therefore$ every element of A is in B.

$\because B \subseteq C \therefore$ every element of B is in C.

Now, each element of A is in B and each element of B is, in its turn, in C.

$\therefore$ Every element of A is in C

$\therefore A \subseteq C$.

Because of these three properties, the subset relationship is said to be " partially ordered. "

1·16 Set of sets

A set, of which each element is itself a set is called a set of sets

e.g., let A={roses, lilies, daffodils}. Then the element 'roses' is itself a set of all roses and similarly 'lilies' 'daffodils' are themselves sets. Thus the set A of all these sets of flowers is a set of sets.

Consider similarly the set $B = \left\{ \{1, 2\}, \phi\ \{3, 4, 5\}. \right\}$ Here, $\{1, 2\}$ is itself a set, and it occurs in B as an element. Similarly ϕ, $\{3, 4, 5, \}$ are each a set. Hence B is a set of sets.

1·17 Difference between o, $\{o\}$, ϕ, and $\{\phi\}$:—

o is not a set. It is a real number.

$\{o\}$ is a singleton set containing the only number o.

ϕ is a set containing no element.

$\{\phi\}$ is a set containing the only element. ϕ.

***1·18 Power set :** Given any set A, it has a number of subsets. A set can be constructed with all these subsets occuring in it as elements. Such a set of all the subsets of A is called the power set of A and is denoted by $\mathbb{P}(A)$ or 2^A

e. g., (i) Let $A = \{ \alpha \}$ then the subsets of A are $\emptyset$ and $\{ \alpha \}$.

Therefore, $P(A) = \{ \phi, \{ \alpha \} \}$

(ii) Let $K = \{ 1, 2 \}$

The subsets of K are $\emptyset$, $\{ 1 \}$, $\{ 2 \}$, $\{ 1, 2. \}$

Therefore, $P(K) = \{ \emptyset, \{ 1 \}, \{ 2 \}, \{ 1, 2 \} \}$

1·19 Comparable sets :

When two sets A and B are such that one of them is a subset of the other, they are said to be comparable sets.

For example,

(1) $A = \{ 2, 3, 4 \}$, $B = \{ 10, 2, 11, 3, 4, 9 \}$ are comparable sets. For, $A \subseteq B$.

(2) Let G be the set of red roses and H be the set of all roses. Then G and H are comparable sets as $G \subseteq H$.

If two sets are not comparable, then they are said to be "not comparable" or "non-comparable" and it is written in short as 'nc'. The case of disjoint sets is a particular case of 'nc'.

For example,

(i) Let A be the set of boys in the hockey team of a college and B be the same in the football team of the same college.
Then the sets A and B are nc.

(ii) Let $\alpha = \{ 0, 1, 2, 3 \}$ and $\beta = \{ 8, 9, 1 \}$

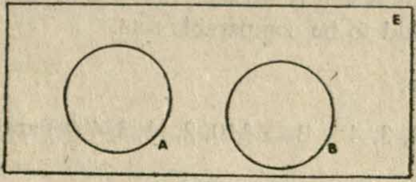
Then the sets α and β are nc sets.

1·20 Universal set :

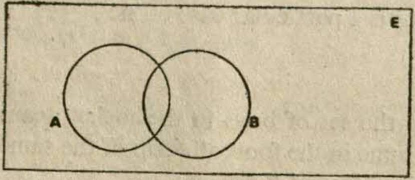
In any discussion of sets, it is always possible to construct a superset of all the sets under consideration. Such a superset is called "Universal set" or "Universe of discourse". A universal set may differ from context to context and will be denoted by E.

1·21 Venn—Euler diagrams

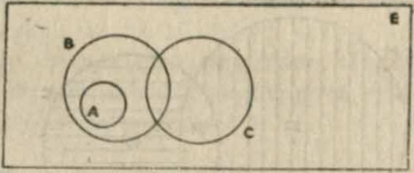
Set theoretical operations can be illustrated by means of the so-called Venn-Euler diagrams or simply, Venn diagrams. In any discussion of sets, the universal set is represented by a bounded area say a rectangle, and all the other sets under consideration by bounded areas (generally but not necessarily, circles) within the rectangle.
e. g., (a) Two disjoint sets A and B can be represented as:



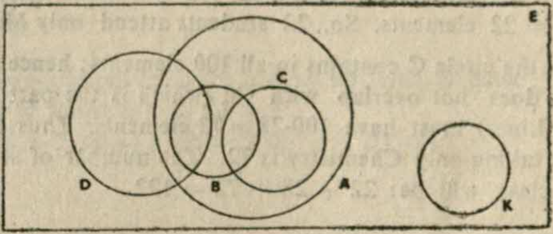
(b) If the sets A and B are 'nc' and not disjoint, then they can be represented as :



(c) If $A \subseteq B$, C is nc but not disjoint with B and C is disjoint from A, then the diagram can be drawn as :



(d) If A ⊇ B, C ⊆ A, B and C are nc but not disjoint, D is nc but not disjoint with A and also with B, but is disjoint from C K is a set disjoint to each of the sets A, B, C, D then the Venn diagram can be drawn as :



Venn diagrams are very useful in solving various types of problems of which the following are two illustrations.

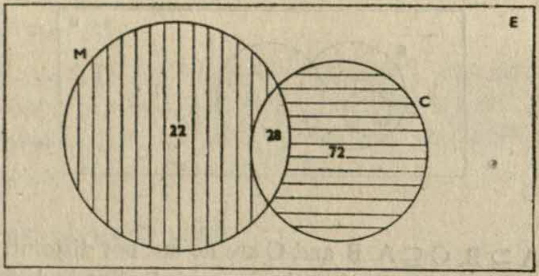
Example 1. In a certain school, it is found that 50 students attend Mathematics class, 100 students attend Chemistry class, and 28 students attend both the classes.

(i) How many students attend Mathematics but not Chemistry?

(ii) How many attend Chemistry but not Mathematics?

(iii) How many students will there be in a combined class of these two subjects?

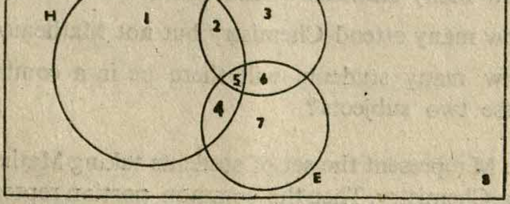
Let the circle M represent the set of students taking Mathematics and C the set taking Chemistry. Then the common portion represents students



taking both the subjects, and hence must have 28 elements. The circle M itself contains 50 elements; hence the portion of M which does not overlap with C (which is the portion shaded by vertical lines) must contain 50-28 = 22 elements. So, 22 students attend only Mathematics.

Similarly, the circle C contains in all 100 elements; hence the portion of C which does not overlap with M (which is the part marked by horizontal Lines) must have 100-28 = 72 elements. Thus, the number of students taking only Chemistry is 72. The number of students in a combined class will be: 22 + 28 + 72 = 122.

Example 2. In a certain school, 28 students take Hindi, 23 students take Marathi and 23 take English, 12 take Hindi and Marathi, 11 take Hindi and English and 8 take Marathi and English. If 5 students take all three languages and 9 take none at all, find the number of students taking (i) Marathi only, (ii) Hindi only, (iii) English only (iv) Hindi but not Marathi; also find (v) the total number of students in the school.



Let in the above figure,the rectangle represent the universal set and the circles H, M, E represent the set of students taking Hindi, Marathi and English respectively. Denote each independent closed part by one and only one symbol. We can now express the given conditions in terms of equations made up of these symbols.

Thus, ① + ② + ④ + ⑤ = 28
② + ③ + ⑤ + ⑥ = 23
④ + ⑤ + ⑥ + ⑦ = 23
④ + ⑤ = 11
② + ⑤ = 12
⑤ + ⑥ = 8
⑤ = 5
⑧ = 9

Solving these equations from the bottom upwards, we have,
⑤ = 5 and ⑤ + ⑥ = 8 ∴ ⑥ = 3
Similarly, ② = 7, ④ = 6, ⑦ = 9, ③ = 8, ① = 10.

Therefore,

(i) No. of students taking Marathi only and no other language = ③ = 8

(ii) No. of students taking Hindi only = ① = 10.

(iii) No. of students taking English only = ⑦ = 9

(iv) No. of students taking Hindi but not Marathi
= ① + ④ = 10 + 6 = 16.

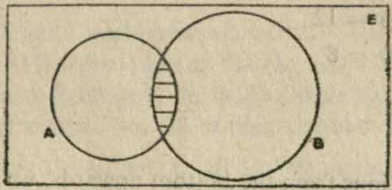
(v) Total number of students in the school
= ① + ② + ③ + ④ + ⑤ + ⑥ + ⑦ + ⑧
= 10 + 7 + 8 + 6 + 5 + 3 + 9 + 9
= 57.

1·22 Algebra of sets :

Just as there are operations of addition, multplication, subrtaction and division which combine any two real numbers to produce another real number, we now define some operations in sets which combine sets (two at a time) to produce another set contained in the universe of discourse.

1. Intersection:

If A and B are any two sets, then the intersection of A and B, written as $A \cap B$ (read "A intersection B") is the set of all elements which are in A and also in B.



Thus, $A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$. In the Venn diagram drawn above the shaded portion represents $A \cap B$.

e.g., (i) If $A = \{ 3, 4, 9, 10, 13 \}$ and $B = \{ 5, 8, 4, 9, 3 \}$
then $A \cap B = \{ 3, 4, 9 \}$

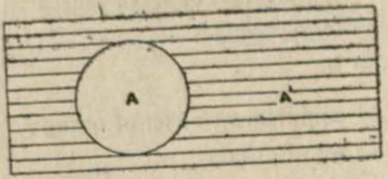
(ii) If A = { John, Henry, Mary, Jean }
B = { Mary, Jean, Ruth, Elsie }
Then $A \cap B$ = { Mary, Jean }

(iii) Let G be the set of boys in standard VII of a school and H be the set of boys in standard VIII of the same school.
Then $G \cap H = \phi$, because G and H do not have any common element.

(iv) Let X be the set of boys in Hislop College
Y be the set of boys in Science College.
Then $X \cap Y = \phi$.

2. Complement or prime:

The complement or prime of a set A is the set of all those elements of the universal set E which do not occur in A. Complement of a set A is denoted by A′. In the Venn diagram drawn below, the shaded portion represents the complement of A.



Symbollically, $A' = \{ x \mid x \in E \text{ and } x \notin A \}$

Examples:

(i) Let $E = \{ 2, 0, 3, 9, 6 \}$, $A = \{ 2, 9 \}$, $B = \{ 6, 3, 2 \}$
Then $A' = \{0, 3, 6\}$ and $B' = \{0, 9\}$

(ii) Let a basket contain oranges and grapes, and let E denote the set of all the fruits in the basket, B denote the set of oranges in it and C denote the set of grapes in it.
Then B′ = Set of grapes in the basket;
and C′ = Set of oranges in the basket.

(iii) Let the universe of discourse be the set of all students in a class, A be the set of boys in the class and B be the set of girls in the class.

Then, A′ = Set of all girls in the class;
and B′ = Set of all boys in the class.

Note that

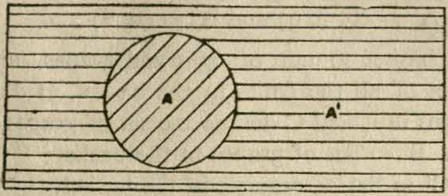
In ex. (i), $(A')' = \text{complement of } A'$
$= \text{complement of } \{ 0, 3, 6 \}$
$= \{ 2, 9 \}$
$= A.$

$$\begin{aligned}\text{and } (B')' &= \text{complement of } B' \\ &= \text{complement of } \{0, 9\} \\ &= \{6, 3, 2\} \\ &= B;\end{aligned}$$

$$\begin{aligned}\text{In ex, (ii) } (B')' &= \text{complement of set of grapes} \\ &= \text{set of oranges} \\ &= B.\end{aligned}$$

$$\begin{aligned}\text{and } (C')' &= \text{complement of set of oranges} \\ &= \text{set of grapes} \\ &= C.\end{aligned}$$

Similarly in ex (iii), $(A')' = A$ and $(B')' = B$.
In general, for any set A, $(A')' = A$.
This can also be seen from the Venn diagram.



A is the set marked with oblique lines, A′ is the set marked with horizontal lines.

$$\begin{aligned}\therefore (A')' &= (\text{Set marked with horizontal lines})' \\ &= \text{Set marked with oblique lines.} \\ &= A.\end{aligned}$$

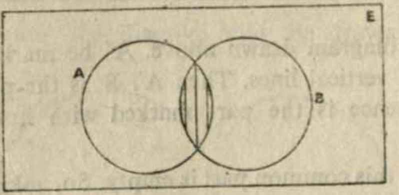
* 1·21 Boolean relations :

Having defined the two operations of intersection and prime, let us now make an attempt to express problems given in ordinary language, in terms of sets alone. The following are a few examples of how a set equation can be interpreted in terms of ordinary language and conversely.

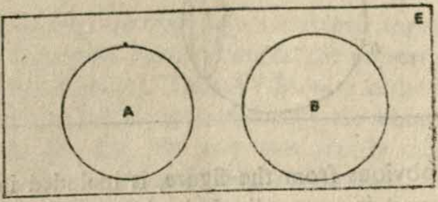
1. $A \cap B = \phi$.

We know that $A \cap B$ is the set of elements common to A and B.

$\therefore$ $A \cap B = \phi$ means this set of common elements is empty. Hence there is no element which is present in A and also in B. The shaded



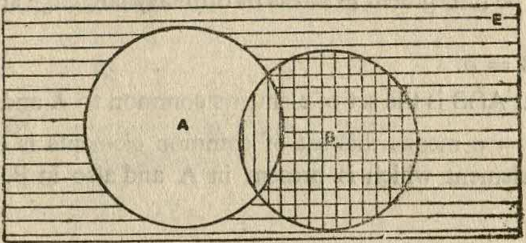
part in the diagram drawn above is, therefore, void. The sets A and B are disjoint sets, and the diagram may be drawn as under:



Thus, $A \cap B = \phi$ is the set language for the words "A and B are disjoint sets". So, whenever we come across the equation $A \cap B = \phi$, we shall at once know that it means the sets A and B are disjoint and, conversely, if we wish to express the fact that A and B are disjoint sets, then all we need to write is $A \cap B = \phi$.

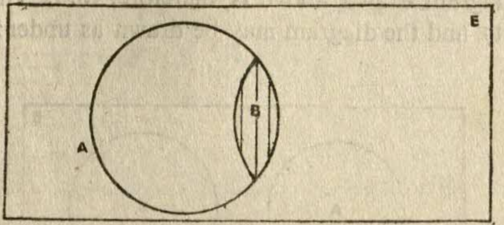
Since the equation $A \cap B = \phi$ signifies the mutual exclusiveness of the two sets, it is said to be a case of "*Mutual exclusion*".

2. $A' \cap B = \phi$



Let in the Venn diagram drawn above, A' be marked with horizontal lines, and B with vertical lines. Then $A' \cap B$ is the portion common to A' and B and hence is the part marked with horizontal as well as vertical lines.

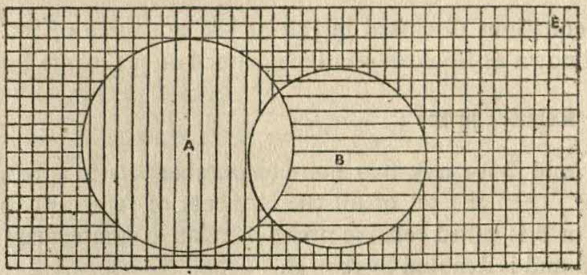
$A' \cap B = \phi$ means this common part is empty. So, taking it off the figure, the diagram may be drawn as shown below.



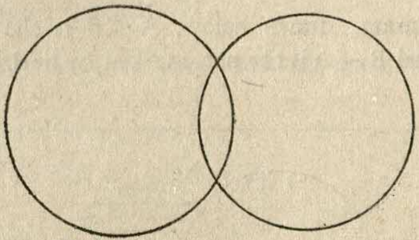
The set B, as is obvious from the figure, is included in the set A. The equation $A' \cap B = \phi$ is, accordingly, said to represent the case of "*Inclusion*". From the discussion made above, it is clear that $A' \cap B = \phi$ is the set language for "B is included in A" or, equivalently, for "B is a subset of A".

3. $A' \cap B' = \phi$.

Let in the Venn diagram drawn on top of page 19, A' represent the set marked with horizontal lines and B' represent the set marked with vertical lines. Then $A' \cap B'$ is the set marked with both horizontal and verical lines. $A' \cap B' = \phi$ means this common part is empty. So,



taking it off the figure, the diagram may be drawn as shown below.



Obviously, it shows that the two sets, combined together, occupy the whole universe of discourse. In other words, the universal set is made up of the two sets taken together. Thus $A' \cap B' = \phi$ is the set language for the words "The sets A and B together occupy the whole universe of discourse." Since, in this case, the two sets jointly exhaust the whole universe, it is known as the case of "*Joint Exhaustion*".

e. g., Let $A = \{2, 5\}$, $B = \{3, 6, 9\}$

If the universal set is made up of these two sets only, we have

$$E = \{2, 5, 3, 6, 9\}$$

Now, $A' = \{3, 6, 9\}$, $B' = \{2, 5\}$, so that A′ and B′ are disjoint.

$$\therefore A' \cap B' = \phi.$$

Thus, this equation is formed when the sets A and B, taken together, make up the universal set.

The three equations discussed in this article are three of the twelve set equations known as Boolean relations, named after George Boole.

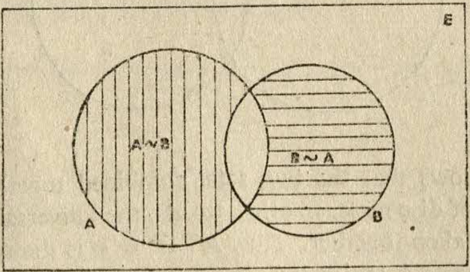
1·22 Relative prime :

Let A and B be any two non-comparable sets in the universe of discourse E. Then the set of all the elements of A except those which occur also in B is called relative prime or relative complement of A with respect to B and is denoted by $A \sim B$.

Thus, $A \sim B = \{x \mid x \in A \text{ and } x \notin B\}$

Similarly, $B \sim A = \{y \mid y \in B \text{ and } y \notin A\}$.

In the Venn diagram, drawn below, $A \sim B$ is the portion marked by vertical lines and $B \sim A$ is the part marked by horizontal lines.



Note that $A \sim B \neq B \sim A$ except when $A = B$.

Example:

Let $A = \{ 3, 4, 9, 5 \}$

$B = \{ 0, 4, 3, 2, 7 \}$

Then $A \sim B = \{ 9, 5 \}$

and $B \sim A = \{ 0, 2, 7 \}$

Note that for any set A,

$E \sim A = A'$ where E is the universal set.

EXERCISE

1. Write in set notations :
   - (i) R is a superset of T.
   - (ii) x is a member of Y.
   - (iii) H is not a subset of S.
   - (iv) Z does not belong to A.
   - (v) B is included in F.
   - (vi) r belongs to A.
   - (vii) S is not a superset of L.

2. State whether each of these statements is correct or incorrect.
   - (i) $\{1, 4, 3\} = \{3, 4, 1\}$
   - (ii) $\{1, 3, 1, 2, 3, 2\} \subseteq \{1, 2, 3\}$
   - (iii) $\{4\} \subseteq \left\{\{4\}\right\}$
   - (iv) $\phi \subseteq \left\{\{4\}\right\}$
   - (v) $\emptyset \in \{4\}$
   - (vi) $\emptyset \subseteq \{4\}$
   - (vii) $6 \in \{S, H, 6, r\}$
   - (viii) $6 \subseteq \{S, H, 6, r\}$
   - (ix) $\{\triangle\} \in \{\triangle, \{\triangle\}, \nabla, \Diamond\}$
   - (x) $\{\triangle\} \subseteq \{\triangle, \{\triangle\}, \nabla, \Diamond\}$
   - (xi) $\{\nabla, \Diamond\} \in \{\triangle, \{\Diamond\}, \nabla, \Diamond\}$

3. Write the following in tabular form:
   - (i) A = { x | x is a vowel in the word "Mathematics". }

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(ii) $B = \{x \mid x^2 - 5x + 6 = 0\}$

(iii) $C = \{y \mid y + 8 = y\,.\}$

(iv) $D = \{m \mid m \text{ is an integer and } m\text{-}3 = 0\}$

(v) $E = \{p \mid p \text{ is an integer and } -2 \leqslant p < 8\}$

(vi) $F = \{t \mid t \text{ is a letter in the word "follow"}\}$

(vii) $G = \{x \mid x \text{ is a digit in the number } 2324.\}$

4. Write in set builder form:

(i) $A = \{s, t, i, r\}$

(ii) $B = \{10, 20, 30, 40 \ldots\ldots\}$

(iii) $C = \{2, 4, 6, 8 \ldots\ldots\}$

(iv) $D = \{6, 7, 8, 9\}$

(v) $E = \{2, -2\}$

5. State whether the following are true or false:

(i) If $A = \left\{\{\triangle, \square\}, \psi\right\}$ and $B = \{\square, \triangle, \psi\}$.

then $B \subseteq A$, $\psi \in A$, $A \subseteq B$, $\triangle \in B$, $\{\square, \triangle\} \in A$, $B = A$.

(ii) $0 \in \phi$.

(iii) $\{0\} \subseteq \phi$

(iv) $\{3, 4\} \in \left\{ \{3, 4\}, \{5, 6\} \right\}$

(v) $5 \in \left\{ 5, \{5\}, 9, 0 \right\}$

6. Draw Venn diagrams for three non-empty sets A, B and C so that

(i) $A \subseteq B \subseteq C$.

(ii) $B \subseteq C$, A intersects C but is disjoint to B.

(iii) Each is disjoint to the other two.

(iv) B and C are disjoint, but A intersects both.

(v) Each intersects the other two.

(vi) $C \subseteq A$, $B \subseteq A$ but B and C are disjoint.

(vii) $A \subseteq B$ and $A \subseteq C$, B and C intersect.

7. Let $A = \{2,3,4\}$, $B = \{x \mid x^2 = 4, x \text{ is } +\text{ve}\}$
$C = \{x \mid x^2 - 6x + 8 = 0\}$; $D = \{x \mid x \text{ is even}\}$.

Complete the following statements by inserting $\subseteq$, $\supseteq$, 'nc' between each pair of sets.

(i) A —— B (ii) A —— C (iii) B —— C
(iv) A —— D (v) B —— D (vi) C —— D.

8. In a certain college, 600 students take Mathematics and 300 take Physics. 173 students are enrolled in both the subjects. In all how many different students are enrolled in the two courses?

9. In a French class there are 30 students and in a German class there are 40 students. How many students will there be in a combined picnic if

(i) the two classes are usually held at different times and 9 students attend both.

(ii) the two classes are usually held at the same time, in different rooms.

10. In the first year class of a women's college 36 students take Philosophy, 23 students take Economics, and 13 students take Sanskrit. 6 students take Philosophy and Economics, 11 students take Philosophy and Sanskrit, 4 students take Economics and Sanskrit, while 1 student takes all the three courses.
Show that the data given above are inconsistent.

11. In the Pre-University class of a certain college 18 students take Physics, 23 students take Chemistry, and 24 students take Mathematics. Out of these, 14 students take both Chemistry and Mathematics, 12 take both Physics and Chemistry, 11 take both Mathematics and Physics, and 5 take all the three subjects. If 10 students do not offer any one of the 3 subjects, find

    (i) How many students take Mathematics but not Chemistry.
    (ii) How many students take Mathematics but not Chemistry or Physics.
    (iii) How many students there are in the class.

12. State whether each of the following sets are finite or infinite:

    (i) the set of numbers which are divisible by 3;
    (ii) the set of cirlces passing through a fixed point;
    (iii) the set of straight lines intersecting a given line at a given point;
    (iv) the set of all living males on the earth;
    (v) the set of concentric circles having a fixed centre;
    (iv) the set of all real numbers statisfying the equation
    $$(x^2-5x+6)\ (3x^2-7x-20)=0;$$
    (vii) the set of points in a given plane at a unit distance from a fixed point in the plane;
    (viii) the set of points on a given straight line at a unit distance from a given point on the line.

13. Let PQ and RS be any two straight lines and let A be the set of points in PQ and B the set of points in RS.

    Fill in the blank A —B by "disjoint" "nc", $\subseteq$, $\supseteq$ as required in the following cases :

    (i) PQ and RS are parallel lines.
    (ii) P and Q lie on opposite sides of RS.
    (iii) P and Q lie on the line RS, between R and S.

(iv) PQ and RS coincide and are equal in length.

(v) R and S lie on the line PQ between P and Q.

ANSWERS

1. (i) $R \supseteq T$; (ii) $x \in y$; (iii) $H \nsubseteq S$; (iv) $Z \notin A$; (v) $B \subseteq F$; (vi) $r \in A$; (vii) $S \nsupseteq L$.

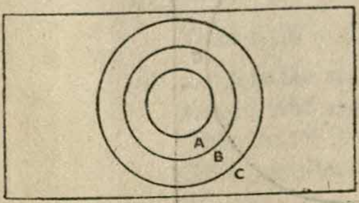
2. (i) Correct, (ii) Correct, (iii) Incorrect, (iv) Correct, (v) Incorrect, (vi) Correct, (vii) Correct, (viii) Incorrect, (ix) Correct, (x) Correct, (xi) Incorrect.

3. (i) $\{a, e, i\}$, (ii) $\{3, 2\}$, (iii) ϕ, (iv) $\{3\}$, (v) $\{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$, (vi) $\{f, 1, 0, w\}$, (vii) $\{2, 3, 4\}$

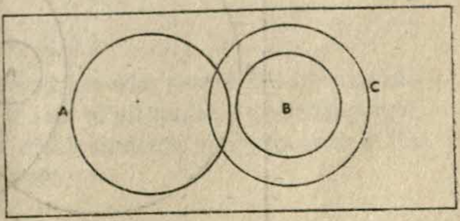
4. (i) {x | x is a letter in the word 'Stir'}
   (ii) {x | x is a positive integral multiple of 10.}
   (iii) {x | x is a +ve even Integer.}
   (iv) $\{x \mid 6 \leqslant x \leqslant 9$ and x is an Integer. $\}$.
   (v) $\{x \mid x^2 - 4 = 0\}$

5. (i) False, True, False, True, True, False.
   (ii) False, (iii) False, (iv) True, (v) True.

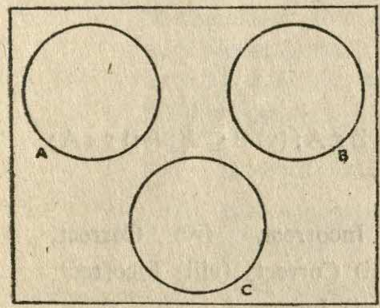
6 (i)



(ii)



(iii)



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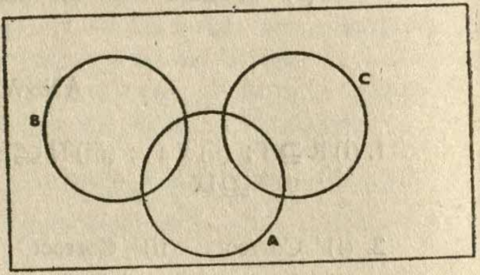

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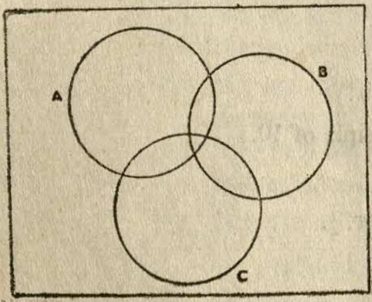

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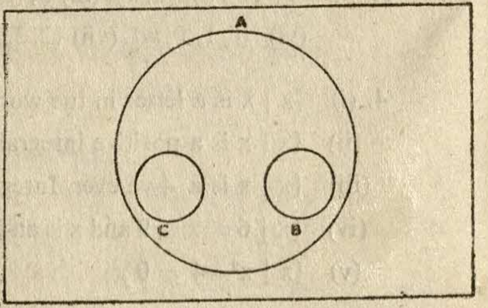
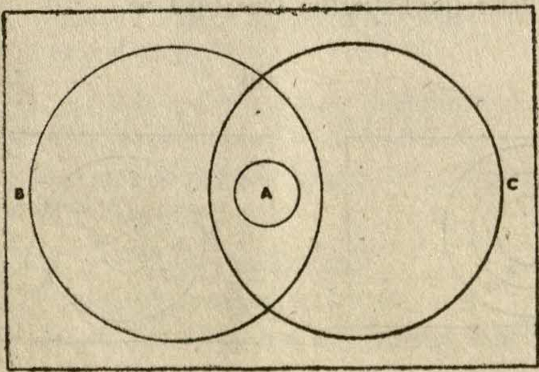


(vii)



7. (i) $\supseteq$, (ii) $\supseteq$, (iii) $\subseteq$, (iv) nc, (v) $\subseteq$, (vi) $\subseteq$

8. 727, 9. 61, 70, 11. (i) 10, (ii) 4, (iii) 43

12. (i) Infinite, (ii) Infinite, (iii) Infinite, (iv) Finite.
(v) Infinite, (vi) Finite, (vii) Infinite, (viii) Finite.

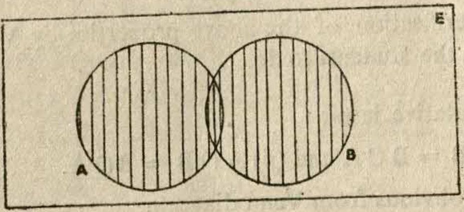
13. (i) Disjoint, (ii) nc, (iii) $\subseteq$, (iv) =, (v) $\supseteq$.

1·23 Union of sets

The union of two sets A and B written as A∪B, is the set of all the elements of A, together with all the elements of B. That is, if an element is in A or in B, or in both, then it will be in A∪B.

Thus, $A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$

In the Venn diagram drawn below, the shaded portion represents the set A∪B.



Illustrations:

(i) Let $A = \{ 9, 6, 4 \}$, $B = \{ 3, 4, 5 \}$

Then $A \cup B = \{ 9, 6, 4, 3, 5 \}$

(ii) Let A be the set of all students who passed S.S.C. examination in 1966 and B be the set of all students who failed in it. Then A∪B is the set of all students who appeared in the S.S.C. examination in 1966.

(iii) Let $A = \{ +, \div, \psi, \propto \}$

$B = \{3, m, 9, \propto\}$

Then $A \cup B = \{+, \div, \psi, \propto, 3, m, 9\}$

1·24 Properties of union, intersection, prime :

1. $(A')' = A$

2. $A \cup A' = E$ and $A \cap A' = \phi$

3. $E' = \phi$ and $\phi' = E$

4. (a) $A \cup \phi = A$, (b) $A \cup E = E$, (c) $A \cap E = A$, (d) $A \cap \phi = \phi$
   (a) and (c) are called the identity laws.

5. $A \cup A = A$ and $A \cap A = A$

   These are called the idempotent laws.

6. $A \cap B = A$ if and only if $A \subseteq B$

   $A \cup B = A$ if and only if $B \subseteq A$.

   The verification of the above properties by Venn diagrams is left for the students to do.

7. Commutative laws:

   (i) $A \cup B = B \cup A$ and (ii) $A \cap B = B \cap A$.

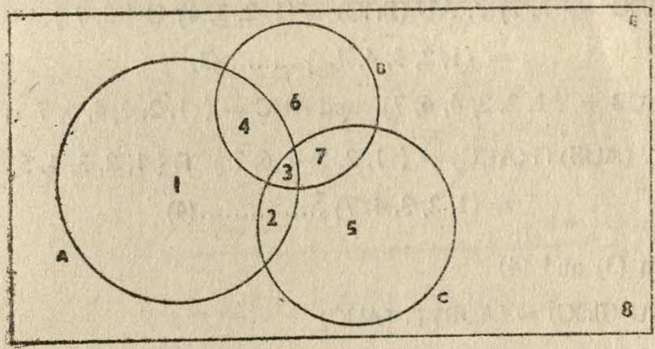
   This is obvious from Venn diagram.

8. Associative laws:

   (a) Associative law for union: $A \cup (B \cup C) = (A \cup B) \cup C$.

   (b) Associative law for intersection: $A \cap (B \cap C) = (A \cap B) \cap C$.
   Let A. B. C. be any three sets as shown in the figure and E be the universal set. Represent each independent, bounded region by one and only one symbol.

   Thus, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 6, 7\}$,
   $C = \{2, 3, 5, 7\}$ and $E = \{1, 2, 3, 4, 5, 6, 7, 8\}$



Proof:

(a) $B \cup C = \{2, 3, 4, 5, 6, 7,\}$

$\therefore A \cup (B \cup C) = \{1, 2, 3, 4\} \cup \{2, 3, 4, 5, 6, 7,\}$

$= \{1, 2, 3, 4, 5, 6, 7\}$ .... (1)

$A \cup B = \{1, 2, 3, 4, 6, 7\}$

$\therefore (A \cup B) \cup C = \{1, 2, 3, 4, 6, 7\} \cup \{2, 3, 5, 7\}$

$= \{1, 2, 3, 4, 5, 6, 7\}$........................... (2)

From (1) and (2), $A \cup (B \cup C) = (A \cup B) \cup C$.

(b) $B \cap C = \{3, 7\}$ $\therefore A \cap (B \cap C) = \{1, 2, 3, 4\} \cap \{3, 7\} = \{3\}$

$A \cap B = \{3, 4\}$ $\therefore (A \cap B) \cap C = \{3, 4\} \cap \{2, 3, 5, 7\} = \{3\}$

Therefore, $A \cap (B \cap C) = (A \cap B) \cap C$.

9. Distributive laws:

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

This is the distribution of union over intersection.

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

This is the distribution of intersection over union.

Proof:

(i) Referring to the same figure drawn above,

$B \cap C = \{3, 7\}$ $\therefore$ $A \cup (B \cap C) = \{1, 2, 3, 4\} \cup \{3, 7\}$

$= \{1, 2, 3, 4, 7, \}$ ....(3)

$A \cup B = \{1, 2, 3, 4, 6, 7\}$ and $A \cup C = \{1, 2, 3, 4, 5, 7\}$

$\therefore (A \cup B) \cap (A \cup C) = \{1, 2, 3, 4, 6, 7\} \cap \{1, 2, 3, 4, 5, 7\}$

$= \{1, 2, 3, 4\ 7\}$ ....(4)

from (3) and (4)

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) In the same figure.

$B \cup C = \{2. 3, 4, 5, 6, 7\}$

$\therefore A \cap (B \cup C) = \{1, 2, 3, 4\} \cap \{2, 3, 4, 5, 6, 7\}$

$= \{2, 3, 4\}$

$A \cap B = \{3, 4\}$ and $A \cap C = \{2, 3\}$

$\therefore (A \cap B) \cup (A \cap C) = \{3, 4\} \cup \{2, 3\} = \{2, 3, 4\}$

$\therefore A \cap (B \cap C) = (A \cap B) \cup (A \cap C)$

9. D' Morgan's laws:

(i) $(A \cap B)' = A' \cup B'$

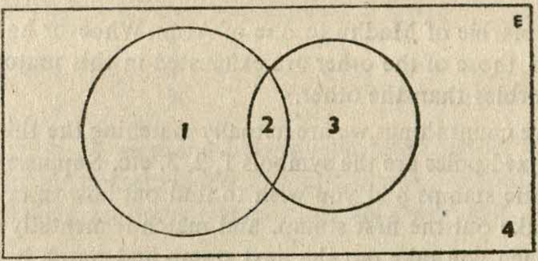
(ii) $(A \cup B)' = A' \cap B'$

The first law states that "The complement of the intersection of two sets is equal to the union of their complements."

The second law states that "The complement of the union of two sets is equal to the intersection of their complements."

Proof:

Let, in the figure drawn below, E be the universal set and A and B be any two sets contained in it. Denote each independent region as marked in the figure.



$\therefore$ $E = \{1, 2, 3, 4\}$; $A = \{1, 2\}$; $B = \{2, 3\}$

(i) $A \cap B = \{2\}$

$\therefore (A \cap B)' = \{1, 3, 4\}$

also, $A' = \{3, 4\}$ and $B' = \{1, 4\}$

$\therefore A' \cup B' = \{1, 3, 4\}$

$\therefore (A \cap B)' = A' \cup B'$.

(ii) $A \cup B = \{1, 2, 3\}$

$\therefore (A \cup B)' = \{4\}$.

Also, $A' = \{3, 4\}$ and $B' = \{1, 4\}$

$A' \cap B' = \{4\}$

$\therefore (A \cup B)' = A' \cap B'$.

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1·25 Cardinal number and Equivalence classes. :

A shepherd boy who did not know any counting, used to take his cows out for grazing every morning, and bring them back in the evening and tie them to the poles he had fixed for them, one for each cow. One evening he found that one pole was left out extra. He at once knew that one cow was missing. Even though he could not count, he matched the poles to the cows one to one, and thus could have an estimate of his cows comparing them with the poles. The process of matching thus comes out very useful where counting is unknown.

Take another example. Suppose two kids Madhu and Arun who are too young to count, possess some marbles each and wish to find out who has more. The only way for them is to match their marbles, by pairing one marble of Madhu to one of Arun. Whoever has any marble left out after those of the other are exhausted in this matching process, has more marbles than the other.

When we count things, we are actually matching the things mentally. For us, the fixed poles are the symbols 1, 2, 3, etc. Suppose, for example, you have some stamps and you wish to find out how many stamps you have. You take out the first stamp, and match it mentally with the first symbol, 1; then you take out the next stamp and match it with the next fixed symbol, 2. You go on doing this till your stamps are exhausted. If the last fixed symbol you match your last stamp with is 9, then you say you have 9 stamps. Thus, counting is itself a matching, with the advantage that it is a matching to be done mentally and therefore can be done anywhere.

The process of matching is a concept which goes very far into Mathematics, specially when we are unable to count. When are we unable to count? Obviously, when the elements to be counted are so many that the counting process does not seem to come to an end at all. What then, is to be done in such cases?

Before we get an answer to this question, we shall have to discipline the ideas expressed above, and put them in mathematical terms. It is at a much later stage only, that you can be told precisely how to estimate the number of elements of a set which contains so many elements that we are unable to count them. In other words, how to compare the elements of infinite sets. Let us start by making the following considerations.

Consider the sets A = { 2, 3, 5 } and B = { a, b, c }. Suppose a matching is brought about among the elements of the two sets in the following manner.

$$\begin{array}{ccc} 2 & 3 & 5 \\ \updownarrow & \updownarrow & \updownarrow \\ a & b & c \end{array}$$

in which every element of the set A is matched with one and only one element of B. Since the order in which the elements of a set are written is immaterial, matching can be done in many ways, as

$$\begin{array}{cccccccc} 2 & 3 & 5 & & 2 & 3 & 5 & \\ \updownarrow & \updownarrow & \updownarrow & \text{or, as} & \updownarrow & \updownarrow & \updownarrow & \text{etc.} \\ b & c & a & & c & a & b & \end{array}$$

Such an association or mapping between the elements of two sets is called a "one to one correspondence" and the sets are then said to be equivalent. Following are some examples of equivalent sets.

(i) The sets A = { 2, 3 }, B = { John, Harry }, are equivalent for they can be put to a one to one correspondence as

$$\begin{array}{cc} 2 & 3 \\ \updownarrow & \updownarrow \\ \text{John} & \text{Mary} \end{array} \quad \text{or as,} \quad \begin{tikzcd} 2 \arrow[dr, leftrightarrow] & 3 \arrow[dl, leftrightarrow] \\ \text{Mary} & \text{John} \end{tikzcd}$$

(ii) The sets $\{ \theta, \alpha, \beta \}$ and { Gopal, Sita, Rama } are equivalent.

(iii) The set A = { Elsie, Jean, Liza } is not equivalent to the set B = { 7, 9 }, for one of the elements of A will be left out of the association (that is one-one matching) in whatever way it is done.

(iv) Let A be the set of all headmsters in a city and B be the set of all the schools in that city. Then A and B are equivalent, for each school has a headmaster and each headmaster is associated with a school.

1·26 Cardinal number

Consider now the sets A = { 3, 5, 7, }, B = { Govind, Ahmed, Ashok }, C = { Apples, Oranges, Mangoes }, D = { +, ÷, — } and E = { r, m, n }. These sets are such that any two of them can be put to a one to one correspondence. They are all equivalent to each other and share a common property, which is having the same number of elements in each. In fact, the number "3" can be used to designate the number of elements in each set. This number is called the "cardinal number" of the collection of sets A, B, C, D, E. The collection itself is called an "equivalence class" of the sets, and the cardinal number 3 is associated with the equivalence class of A, B, C, D, F.

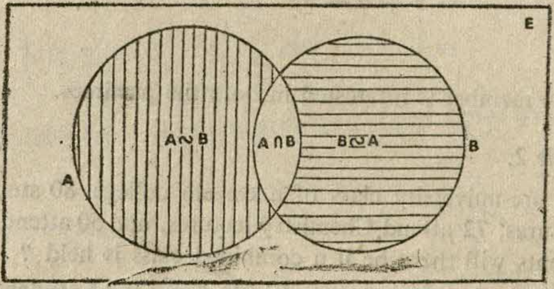
Similarly, the sets A_1, $= \{ \theta, \delta \}$, $B_1 = \{$ Shyam, Arvind $\}$, $C_1 = \{ 9, 0 \}$ are equivalent to each other; hence the set $\{ A_1, B_1, C_1 \}$ is an equivalence class, and since the number of elements in each set of this class is 2, the cardinal number of the class itself is 2.

Since every set of an equivalence class contains the same number of elements, any one of the sets of the class is sufficient to determine the cardinal number. Any set of an equivalence class is, therefore, called a sample element. For example, the number of elements in the set $\{ a, b, \theta, q \}$ is 4. Any set equivalent to this set will also have 4 elements. The equivalence class, formed of all such equivalent sets, must, therefore, have 4 as its cardinal number. Thus $\{ a, b, \theta, q \}$ is the sample element of the equivalence class whose cardinal number is 4.

The cardinal number of a set is simply the cardinal number of the equivalence class containing that set. We shall restrict ourselves to the cardinality of finite sets only, and as has already been observed, the cardinal number of a finite set is the number of elements in that set.

Sample element of equivalence class	Cardinal number
{ □, θ, ▱, ☊ }	4
{ ρ, ψ }	2
{ Madhu, Hari, Dilip }	3
$\{ \} = \phi$	0
{ 20, 0, 9, 7, 6 }	5

1·27 We propose to consider next the relation between the number of elements of any two sets. Suppose we are given that, in a school 170 pupils know German and 220 pupils know Sanskrit. Can we find out how many pupils know only one of the two languages? If it is given that none of the students know both the languages, (that is, the set of pupils knowing German is disjoint from the set of pupils knowing Sanskrit) then the answer would be $170 + 220 = 390$. In general, if A and B are two disjoint sets, then the number of elements in $A \cup B$ is equal to the sum of elements of A and B. The number of elements of a set is usually denoted by n(A). So, if A and B are disjoint sets, then $n(A \cup B) = n(A) + n(B)$. If, however, there are pupils knowing both German and Sanskrit, then the two sets A and B have some elements in common, and the above formula cannot be used for $n(A \cup B)$. We make the following considerations and evolve the general formula as follows:



$A \cup B$ is formed of the three disjoint sets $A \sim B$, $A \cap B$ and $B \sim A$.

$\therefore\ n(A \cup B) = n(A \sim B) + n(B \sim A) + n(A \cap B)$ .... (1)

The set A is formed of the two disjoint sets $A \sim B$ and $A \cap B$.

$\therefore\ n(A) = n(A \sim B) + n(A \cap B)$ .... (2)

Similarly, the set B is formed of the two disjoint sets $A \cap B$ and $B \sim A$.

$\therefore\ n(B) = n(A \cap B) + n(B \sim A)$ .... (3)

Adding (2) and (3),

$n(A) + n(B) = n(A \sim B) + n(B \sim A) + 2\,n(A \cap B)$............ (4)

Substituting from (1) in (4)

$$n(A) + n(B) = n(A \cup B) + n(A \cap B)$$

$$\therefore\ n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

1·28 *Example 1.*

In a family of 8 members, 3 are interested in sports and 6 are interested in reading. Find how many members are interested in sports as well as reading.

Let A denote the set of members interested in sports and B denote the set of members interested in reading. Then, $n(A \cup B) = 8$, $n(A) = 3$, $n(B) = 6$. Using the formula derived above,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$\therefore n(A \cap B) = n(A) + n(B) - n(A \cup B)$$
$$= 3 + 6 - 8$$
$$= 1$$

So, only one member is interested in both the pastimes.

Example 2.

In the pre-university class of a certain college, 80 students attend Physics lectures, 72 attend Chemistry lectures, and 50 attend both. How many students will there be if a combined class is held ?

Let P and C represent respectively the set of students attending Physics and Chemistry classes. Then,

$$n(P) = 80,\ n(C) = 72,\ n(P \cap C) = 50$$
$$n(P \cup C) = n(P) + n(C) - n(P \cap C)$$
$$= 80 + 72 - 50$$
$$= 102$$

Hence the combined class will contain 102 students.

1·29 Partition:

A partition of a set A is a subdivision of the set into non-empty subsets such that

(a) The subsets are pairwise disjoint. That is, no two subsets have any common element.

(b) The union of all the subsets is the set A itself.
For example, let A = {1, 2, 3, 4}. Then the subsets {1}, {2, 3,} {4} are such that they are pairwise disjoint and their union is {1, 2, 3, 4}, that is, the set A itself.

Hence $\Big\{ \{1\}, \{2, 3\}, \{4\} \Big\}$ is a partition of A. Some other partitions of the same set A are

$$\{\{1, 2, 3\},\ \{4\}\}$$

$$\{\{1, 3\},\ \{2, 4\}\};$$

$$\{\{4, 2\},\ \{1\},\ \{3\}\} \text{ etc.}$$

Pictorially, let a set S be represented by a closed figure as in the adjacent diagram.

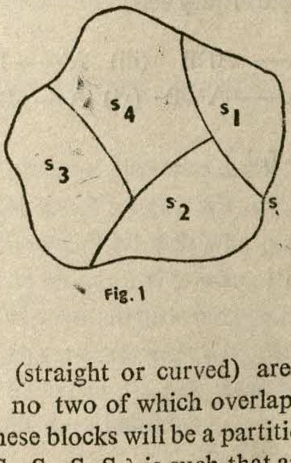


Fig. 1

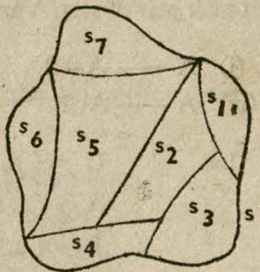


Fig. 2

If lines (straight or curved) are drawn dividing the closed figure into blocks, no two of which overlap, then the collection of sets represented by all these blocks will be a partition of the set S. Thus in Fig. 1, the collection $\{S_1, S_2, S_3\ S_4\}$ is such that any two of its elements are mutually disjoint (i.e. they are pairwise disjoint) and their union is the given set, because $S_1 \cup S_2 \cup S_3 \cup S_4 = S$.

$\therefore\ \{S_1\ S_2, S_3, S_4\}$ is a partition of S.

The same set S can be partitioned as in Fig. 2; so that another partition of S is $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$.

Given a set, therefore, we can make as many different partitions of it as we desire. The union of all the sets forming any one of these partitions will always be the original set.

EXERCISE

1. Draw a Venn diagram in each of the following cases for three non-empty sets A, B and C satisfying the following properties:

(i) $A \subseteq B$ and $C \subseteq B$ and $A \cap C = \phi$

(ii) $A \subseteq C$ and $A \neq C$ and $B \cap C = \phi$

(iii) $A \subseteq B$ and $C \not\subseteq B$ and $A \cap C \neq \phi$

(iv) $A \subseteq B \cap C$ and $B \cap C \neq \phi$.

2. Complete the following statements by inserting $\subseteq$. $\supseteq$, or 'nc' between each pair of sets, A and B being arbitrary sets.

(i) A —— $A \sim B$ (ii) A——$A \cap B$ (iii) A'——$B \sim A$

(iv) A —— $A \cup B$ (v) A'——$A \sim B$ (vi) A——$B \sim A$

3. Let $U = \{ a, c, e, i, n, r, t \}$

$A = \{ t, r, a, i, n \}$

$B = \{ a, r, t \}$

$C = \{ r, e, t, a, i\ n \}$

$D = \{ c, e, n, t \}$

$E = \{ n, i, c, e \}$

U being the universe of discourse, find

(i) A' (ii) B' (iii) $D \cap C$ (iv) $C \cap D$ (v) $D \cap E$ (vi) $A \cap E$ (vii) $B \cup E$ (viii) $C \cup B$ (ix) $(C \cap D) \cap E$ (x) $C \cap (D \cap E)$ (xi) $B \cap E$.

Hence show that $D \cap C = C \cap D$ and $(C \cap D) \cap E = C \cap (D \cap E)$

4. In the previous example, find (i) $B \cup D$, (ii) $D \cup B$ (iii) $B \cup (F \cup C)$ (iv) $(B \cup E) \cup C$ and hence show that $B \cup D = D \cup B$ and $B \cup (E \cup C) = (B \cup E) \cup C$

5. With the sets as given in example 3, fill up the following blanks with $\subseteq$, $\supseteq$, 'nc' :

(i) B——A (ii) B——E (iii) D——E (iv) B—A—C
(v) D′——E′ (vi) B′——A′

6. Let E be the set of all girls in a certain locality, and

P be the set of girls who drink tea.
Q be the set of girls who play hockey.
R be the set of girls who play kho-kho.

Write in ordinary language the following set expressions :—

(i) $P \cap Q$ (ii) P' (iii) Q' (iv) R' (v) $Q \cap R$ (vi) $P \cap R$ (vii) $P' \cap Q$ (viii) $P \cap R'$

7. In the previous example, translate the following into set language :—

(i) The set of girls who drink tea and play kho-kho.
(ii) The set of all girls who drink tea but do not play kho-kho.
(iii) The set of all girls who play hockey and kho-kho.
(iv) The set of all girls who play hockey but not kho-kho.
(v) The set of all girls who do not play hockey but play kho-kho.
(vi) The set of all girls who play neither hockey nor kho-kho.
(vii) The set of all girls who do not drink tea but play hockey and also kho-kho.

8. Let E be the set of all boys in a given high school and

A be the boys who are tall.
B be the boys who are intelligent.
C be the boys who are handsome.

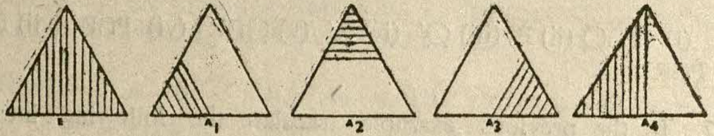
Write the following in ordinary language.

(i) $A \cap B$ (ii) $B \cap C$ (iii) $C \cap A$ (iv) A' (v) B' (vi) C (vii) $(A \cap B) \cap C$ (viii) $A' \cap B'$ (ix) $A' \cap C'$ (x) $A \cup C$.

9. In the previous example, translate the following in set notations:

(i) The set of all boys who are tall and handsome.
(ii) The set of all boys who are tall but not handsome.
(iii) The set of all boys who are handsome but not intelligent.
(iv) The set of all boys who are not tall.
(v) The set of all boys who are not intelligent.
(vi) The set of all boys who are tall, handsome and intelligent.

10. Having given

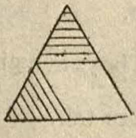


(Universe)

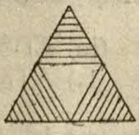
Draw diagrams for the follolwing:

(i) $A_1 \cup A_2$ (ii) $A_2 \cup A_3$ (iii) $A_3 \cup A_4$ (iv) $E \cap A_1$ (v) $E \cap A_2$ (vi) $E \cap A_3$ (vii) $E \cap A_4$ (viii) $A_{.} \cap A_4$ (ix) $A_1 \cap A_4$ (x) A_1' (xi) A_4' (xii) $(A_1 \cup A_2)'$ (xiii) $(A_1 \cup A_2) \cup A_3$ (xiv) $(A_3 \cup A_4)'$ (xv) $(A_2 \cup A_3)'$

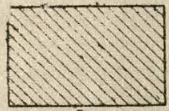
[Hint : $(A_1 \cup A_2)$ is



and hence $(A_1 \cup A_2) \cup A_3$ is,

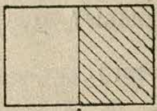


11. Having given.

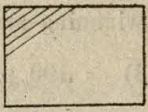


E

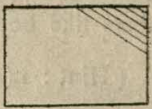
(Universe)



A

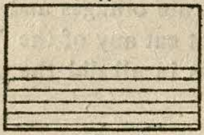


B

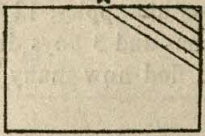


C

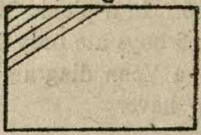
N



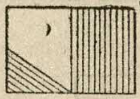
M



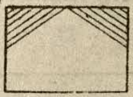
D



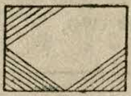
express the following in set notations



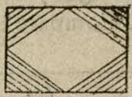
(i)



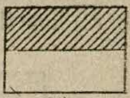
(ii)



(iii)



(iv)



(v)

12. Let I denote the set of all integers and

$$A = \{x \in I \mid 1 < x < 5\}$$
$$B = \{x \in I \mid 2 < x < 7\}$$
$$C = \{x \in I \mid 1 \leqslant x < 6\}$$
$$D = \{x \in I \mid 4 < x \leqslant 8\}$$
$$E = \{x \in I \mid 7 \leqslant x < 10\}$$
$$F = \{x \in I \mid 5 \leqslant x \leqslant 9\}$$

write each of these in a tabular form.

13. In the previous example, find the following :

(i) $A \cap B$, $B \cap C$, $D \cap F$, $C \cap E$, $A \cap F$, $A \cap E$.

(ii) $B \cup E$, $B \cup F$, $C \cup A$, $D \cup F$, $F \cup E$, $C \cup E$, $C \cup F$, $F \cup A$.

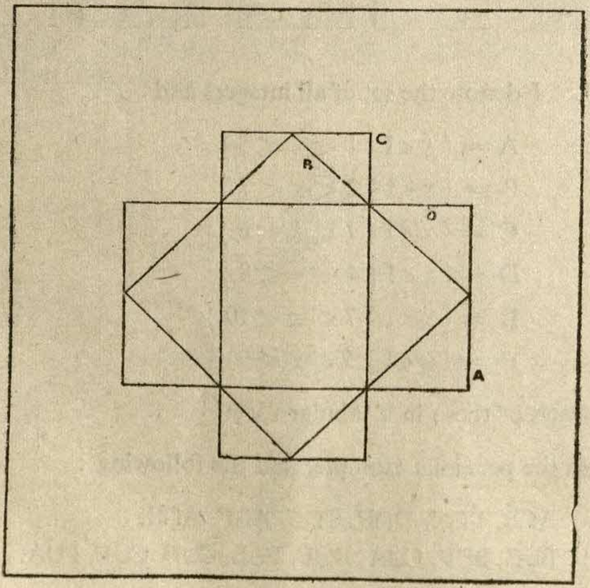
14. In a certain club, 60 percent of the members are fond of swimming and 70 percent are fond of tennis. At least how many of the members like both swimming and tennis?

(Hint : $n(A \cup B) = 100$ percent).

15. In a picnic of 70 boys, some took part in swimming, 60 boys could climb the trees and 5 boys took part in swimming and also climbing the trees. In all, how many boys took part in swimming? Each boy took part at least in swimming or climbing the tree.

16. In a picnic 30 boys ate apples, 12 boys ate oranges and 7 ate both; 5 boys ate only bananas and 3 boys did not eat any of the fruits. Draw a Venn diagram and find how many boys in all did the picnic group have.

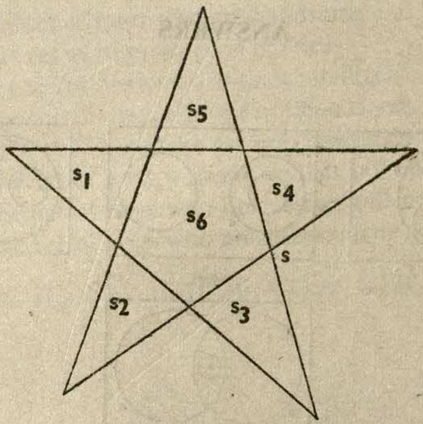
17. Draw two pictures of the following diagram. In one, shade lightly the part $(A \cap B)$, then shade C and then shade heavily $(A \cap B) \cap C$. In the other, shade lightly the part $B \cap C$, then shade A and lastly shade $A \cap (B \cap C)$ heavily. Compare the two.



Is $(A \cap B) \cap C = A \cap (B \cap C)$?

Prove similarly that (i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$; (iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

18. Let set S be represented in the Venn diagram by the star shown below, and the sets S_1, S_2, S_3, S_4, S_5, S_6 are as shown in the figure. Is $\{ S_1, S_2, S_3, S_4, S_5, S_6 \}$ a partition of S? Justify your answer.



19. Given $A = \{ \propto, \beta, a, 2, o, \triangle, t \}$. Write whether each of the following is a partition of A. Justify your answer in each case.

(i) $\left\{ \{ \propto, \beta, 0\}, \{a, 2\}, \{\triangle, t\} \right\}$ (ii) $\left\{ \{2, 0, \triangle, t\}, \{\beta, a\}, \{t, \propto\} \right\}$

(iii) $\left\{ \{\beta, t, 0\}, \{\triangle\}, \{ \propto, 2\} \right\}$ (iv) $\left\{ \{2\}, \{\triangle, 0\}, \{ t, a, \beta, \propto \} \right\}$

(v) $\left\{ \{0\}, \{2,t\}, \{a, 0, \triangle\}, \{ \propto, \beta\} \right\}$

20. For each set on the left, its complement can be located among the sets on the right. The universe of discourse is the set of natural numbers N = 1, 2, 3......

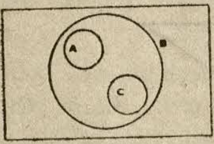
(i) $A = \{ x \in N \mid x < 100 \}$
(ii) $B = \{ x \in N \mid x = 3n,\ n \in N \}$
(iii) $C = \{ x \in N \mid x \leqslant 100 \}$
(iv) $D = \{ x \in N \mid x > 20 \}$
(v) $E = \{ x \in N \mid x$ is a prime number $\}$

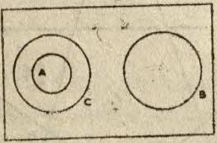
$S = \{ x \in N \mid x \leqslant 20 \}$
$T = \{ x \in N \mid x > 100 \}$
$U = \{ x \in N \mid x \neq 3n,\ n \in N \}$
$V = \{ x \in N \mid x$ is not a prime number $\}$
$W = \{ x \in N \mid x \geqslant 100 \}$

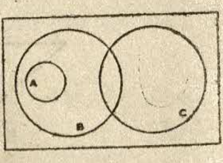
Find out the complement of each set on left from among the sets on right.

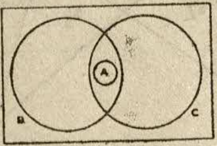
ANSWERS

1.









2. (i) $\supseteq$ (ii) $\supseteq$ (iii) $\supseteq$ (iv) $\subseteq$ (v) nc (vi) nc

3. (i) {c, e} (ii) {c, e, i, n} (iii) {e, n, t} (iv) {e, n, t} (v) {n, c,e}
(vi) { n, i } (vii) { a, r, t, n, i, c, e } (viii) { r, e, t, a, i, n, }
(ix) {n, e} (x) {n, e} (xi) ϕ

4. (i) {a, r, t, c, e, n} (ii) {a, r, t, c,e, n} (iii) { a, r, t, n, i, c, e }
(iv) {a, r, t, n, i, c, e,}

5. (i) $\subseteq$ (ii) nc (iii) nc (vi) $\subseteq$, $\subseteq$ (v) nc (vi) $\supseteq$

6. (i) The set of all girls who drink tea and play hockey.
(ii) The set of girls who do not drink tea.
(iii) The set of girls who do not play hockey.
(iv) The set of girls who do not play kho-kho.
(v) The set of girls who play hockey as well as kho-kho.

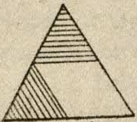
(vi) The set of girls who drink tea and play kho-kho.
(vii) The set of girls who do not drink tea and play hockey.
(viii) The set of girls who drink tea and do not play kho-kho.

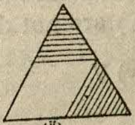
7. (i) $P \cap R$ (ii) $P \cap R'$ (iii) $Q \cap R$ (iv) $Q \cap R'$ (v) $Q' \cap R$ (vi) $Q' \cap R'$ (vii) $P' \cap Q \cap R$.

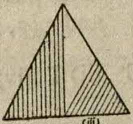
8. (i) The set of boys who are tall and intelligent.
(ii) The set of boys who are intelligent and handsome.
(iii) The set of boys who are handsome and tall.
(iv) The set of boys who are not tall.
(v) The set of boys who are not intelligent.
(vi) The set of boys who are not handsome.
(vii) The set of boys who are tall, intelligent, and handsome.
(viii) The set of boys who are neither tall nor intelligent.
(ix) The set of boys who are neither tall nor handsome.
(x) The set of boys who are either tall or handsome.

9. (i) $A \cap C$ (ii) $A \cap C'$ (iii) $C \cap B'$ (iv) A' (v) B' (vi) $A \cap C \cap B$.

10.


(i)


(ii)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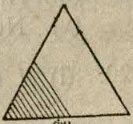
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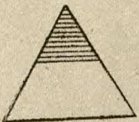
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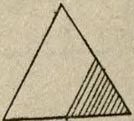
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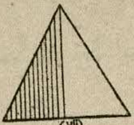
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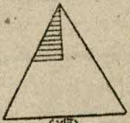
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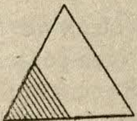
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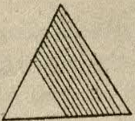
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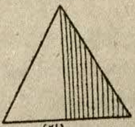
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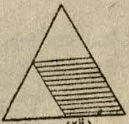
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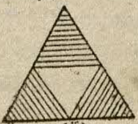
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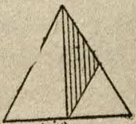
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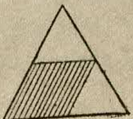

(xv)

11. (i) $A \cup M$ (ii) $B \cup C$ (iii) $B \cup M \cup D$ (iv) $B \cup C \cup D \cup M$ (v) $N \cup D$

12. $A = \{2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{1, 2, 3, 4, 5\}$
$D = \{5, 6, 7, 8\}$, $E = \{7, 8, 9\}$, $F = \{5, 6, 7, 8, 9\}$

13. (i) $\{3, 4\}$, $\{3, 4, 5,\}$, $\{5, 6, 7, 8\}$, ϕ, ϕ, ϕ.
(ii) $\{3, 4, 5, 6, 7, 8, 9\}$, $\{3, 4, 5, 6, 7, 8, 9\}$, $\{1, 2, 3, 4, 5\}$
$\{5, 6, 7, 8, 9\}$, $\{5, 6, 7, 8, 9\}$, $\{1, 2, 3, 4, 5, 7, 8, 9,\}$
$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $\{2, 3, 4, 5, 6, 7, 8, 9.\}$

14. 30 Percent

15. 15.

16. 43

17. Yes.

18. Yes.

19. (i) Yes (ii) No, $\{2, o, \triangle, t\}$ and $\{t, \propto\}$ are not disjoint.
(iii) No, the union does not contain the element 'a', (iv) yes
(v) No, $\{0\}$ and $\{a, o, \triangle\}$ are not disjoint.

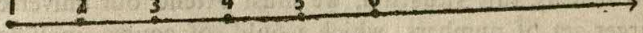
20. (i) W (ii) U (iii) T (iv) S (v) V

CHAPTER TWO

Operations on Sets

2·1 Systems of numbers

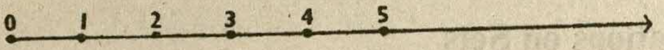
Before men were sophisticated enough to be curious about what exactly a number is, they had words and symbols for numbers and spoke and used them. That the concept of numbers is important there is no controversy about. Like every other important concept, this one has developed and grown through centuries of civilization and scientific progress. When a child learns to count, his starting point is the set 1, 2, 3,4, ... This set, or system of numbers is called the system of "natural numbers" and is denoted by N. It can be represented on a line as:



The Matural numbers (N)

What can we do with these numbers? Of the many uses we make of them, the most common use is to answer the question"how many?"To the question "how many days are there in a week?", the answer is "7", an element of the set of natural numbers. Similarly, "how many states are there in India?", "how many boys are there in this class?", "how many kites do you have?" are questions whose answer is each an element of the set of natural numbers.

Suppose now we put a question "how many men are immortal?" The answer is "none". Can we provide the answer by one of the elements of N? Obviously not. Here the system of natural numbers fails, and it is necessary to extend the system by putting in it another element called "zero" (to represent "none"). The new system is called the system of "counting numbers" and it contains one element namely zero (denoted by 0) more than N.



Counting numbers

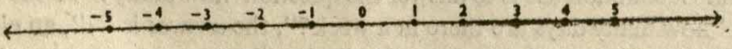
Consider next the question "how many books must be added to 3 books to make it 7?" In other words, what number must be placed in the box of the equation

$$3 + \square = 7$$

to make it true? The answer is "4", and is, like the answer of all the other "how many" questions posed above, an element of the system of counting numbers. Take another question of the same type. What number placed in the box of the equation.

$$3 + \square = 1.$$

will satisfy it? No such number can be found within the number systems we have discussed till now. There is, therefore, no solution of the above equation in terms of the elements in the universe of counting numbers. To remedy this situation, we must extend our universe and invent a larger set of numbers, which would, of course, include the counting numbers too. The new numbers, called traditionally the "negative numbers", are such that each of them added to one (and only one) natural number gives zero as the sum. The number line now looks as:



The integers (1)

This figure presents a more complete look than the previous ones in as much as it is extended indefinitely on both sides. It is not, however, compact. For, between one number and another is a gap having points which do not represent any number in the universe we have.

The system of numbers that we now have containing the natural numbers (also called the positive integers), zero, and the negative integers is called the system of integers or integral numbers and is denoted by I.

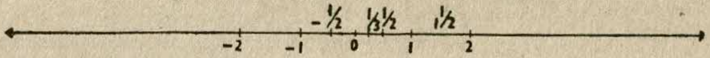
We proceed further still, for this is not sufficient to answer all problems we come across. Consider, for example, the question "what number should be multiplied to 3 to get 15? That is, what number, put into the box, satisfies the equation

$$3 \times \square = 15\ ?$$

The anwer is 5, an element of I. If, however, we wish to fill in the box some number to satisfy the equation

$$4 \times \square = 13$$

there is no number in I which can be used. We must, therefore, extend the universe further. This is done by introducing "fractions" which are parts of an integer. The integers and the fractions together constitute the set of "rational numbers" and this set is denoted by Q. Thus, $N \subset I \subset Q$. The number line is now *dense*, for between any two rational numbers can be inserted many other rational numbers.



This line, full of number points, is strangely enough full of empty spots. In fact, the empty spots outnumber the points.

Consider next the problem "what number, multiplied to itself is 9?" That is, what number should be filled into the box to satisfy the equation

$$\square \times \square = 9\ ?$$

The anwer is 3. (Is there any other number satisfying the same equation?) But if we wish to find a number to satisfy

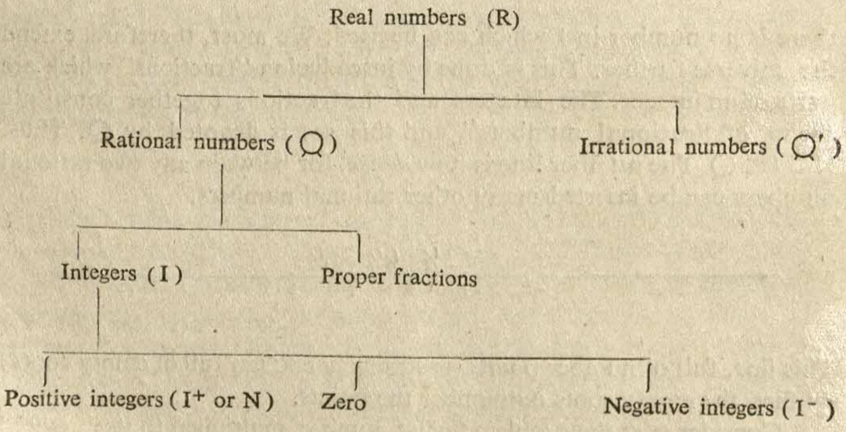
$$\square \times \square = 2$$

we cannot do it with any number in Q. We must extend the number system still further by introducing the system of "irrational numbers" (denoted by Q′) which is such that no number in it can be expressed as the quotient of two integers. For example, $\sqrt{2}$, $\sqrt{13}$, $\sqrt[3]{7}$ is each an irrational number.

The system of all rational numbers together with the irrational numbers is called the sytem of real numbers and is denoted by R. So $Q \subset R$ and $Q' \subset R$.

The number line is now complete. Each point in it represents a real number and each real number can be represented by one (and only one) point in it. A straight line for this reason, is sometimes called a real line. This one-one correspondence between the set of real numbers and the points on a straight line is important.

The systems of numbers we now have is as given below.



2·2 Binary operation :

Having been acquainted with the system of real numbers, we next wish to find out some method by which we may combine two or more elements of a system. To this end, we make the following definition.

Definition : A binary operation $*$ on any set S is a rule which combines any two elements of S to give an element of S.

Illustration :

In N, the set of natural numbers, let * denote the operation of addition. That is, for any two elements a and b of N, $a*b = a+b$. Take any two natural numbers say 7 and 10. Then $7 * 10 = 7 + 10 = 17$

$$\text{and } 17 \in N$$

So, the result obtained by combining 7 and 10 under * is an element of N. This is true for any two elements of N. So * is a binary operation on N.

2·3 Properties of binary operations :

(1) **Closure :** If an operation * defined on a set S is such that it combines any two elements of S in such a way that the resulting element is in S, then the set S is said to be closed under * and the operation * is said to satisfy the property of closure.

According to the definition we have made, a binary operation always satisfies the property of **closure.**

Examples :

1. Let $S = \{-1, 0, 1\}$ and * be the operation of ordinary multiplication.

$$\begin{aligned} \text{Then, } -1 * 0 &= (-1) \times 0 = 0 \quad \text{and} \quad 0 \in S. \\ 0 * 1 &= 0 \times 1 = 0 \quad \text{and} \quad 0 \in S. \\ -1 * 1 &= -1 \times 1 = -1 \text{ and } -1 \in S. \end{aligned}$$

check similarly for $-1 * -1$, $0 * 0$, $1 * 1$.

Thus, whatever two elements of S are combined, they produce an element which is in S. Therefore, S is closed under *.

(2) Let $A = \{0, 1, 2\}$ and * be the operation of ordinary addition.

$$\begin{aligned} \text{Then, } 0 * 1 &= 0 + 1 = 1 \text{ and } 1 \in A. \\ 0 * 2 &= 0 + 2 = 2 \text{ and } 2 \in A. \\ 1 * 2 &= 1 + 2 = 3 \text{ but } 3 \notin A. \end{aligned}$$

So A is not closed under *. Because, when the elements 1 and 2 are combined under *, the resulting element is not in A.

(3) Let $\alpha = \{3, 0, 10, 0\}$, $\gamma = \{4\}$, $\delta = \{3, 0\}$ and ϕ be the null set. Consider $S = \{\alpha, \gamma, \delta, \phi\}$ and let the operation $*$ be defined as the operation of set intersection.

$$\text{Then,}\quad \begin{aligned} \alpha * \gamma &\equiv \alpha \cap \gamma \equiv \emptyset \text{ and } \phi \in S \\ \gamma * \delta &\equiv \gamma \cap \delta \equiv \phi \text{ and } \phi \in S \\ \alpha * \delta &\equiv \alpha \cap \delta \equiv \delta \text{ and } \delta \in S \end{aligned}$$

Test similarly for $\delta * \emptyset$, $\gamma * \phi$ etc. You will find that the resulting element is always in S.

Therefore S is closed under $*$.

(4) In the set N of natural numbers, let $*$ be defined such that if a and b are any two natural numbers, then $a * b = 2a + 3b$. Consider any two natural numbers say 4 and 5.

$$4 * 5 = 2(4) + 3(5) = 23$$
and $23 \in N$.

Thus if 4 and 5 are combined under $*$, the result is an element of N. This is true for any two elements of N. So, N is closed under $*$.

(5) Consider the set $A = \{\alpha, \beta, \gamma\}$ and the operation $\oplus$ defined on it by the following table:

$\oplus$	α	β	γ
α	γ	β	α
β	β	γ	α
γ	α	α	β

$$\text{Then,}\quad \begin{aligned} \alpha \oplus \beta &= \beta \text{ and } \beta \in A \\ \beta \oplus \gamma &= \alpha \text{ and } \alpha \in A \\ \gamma \oplus \alpha &= \alpha \text{ and } \alpha \in A \end{aligned}$$

So, A is closed under $\oplus$.

2·4 (2) Commutative property:

Study the nature of the operation $\oplus$ on A in example 5 above. You will find that $\alpha \oplus \beta = \beta$, $\beta \oplus \alpha = \beta$. Thus $\alpha \oplus \beta = \beta \oplus \alpha$. Similarly, $\beta \oplus \gamma = \gamma \oplus \beta$ and $\alpha \oplus \gamma = \gamma \oplus \alpha$. So, in combining two elements of the set S by $\oplus$, it is immaterial as to which element is the first element and which the second. Such an operation is said to be commutative.

In general, if $*$ is a binary operation defined on a set S such that for any two elements a and b of the set S, $a * b = b * a$, then $*$ is said to be **commutative** in S.

Examples :

(1) The union and intersection of sets are commutative operations; for $A \cup B = B \cup A$ and $A \cap B = B \cap A$ for any two sets A and B.

(2) Let N be the set of natural numbers and $*$ be defined as:

(i) $a * b = a + b$.

So, $b * a = b + a$.

We know that $a + b = b + a$.

$\therefore$ $a * b = b * a$.

$\therefore$ $*$ is commutative.

(ii) $a * b = a + 2b$

So, $b * a = b + 2a$

$\therefore$ $a * b \neq b * a$

$\therefore$ $*$ is not commutative.

(3) (i) In the same set N, let $*$ be defined as ordinary multiplication. Then $*$ is commutative.

(ii) In N, let $*$ be defined as

$a * b = (2a)(b) = 2ab$

Then $b * a = (2b)(a) = 2ba$.

$\therefore a * b = b * a$.

$\therefore$ $*$ is commutative.

(iii) Let $\triangle$ be a binary operation on N such that

$a \triangle b = a\,(a + b)$.

Then $b \triangle a = b\,(b + a)$

$\therefore\ a \triangle b \neq b \triangle a$

$\therefore\ \triangle$ is not commutative.

(4) In the set R of real numbers,

(i) ordinary addition is a commutative operation;

For $a + b = b + a$ for any $a, b \in R$.

(ii) Ordinary multiplication is commutative.

For, $ab = ba$ for any $a, b \in R$.

(iii) Subtraction is not commutative.

Consider the elements 5 and 1 of R.

$5 - 1 = 4$ but $1 - 5 = -4$

$\therefore\ 5 - 1 \neq 1 - 5$

$\therefore$ subtraction is not commutative.

(iv) Division is not commutative.

Consider the elements 9 and 3 of R.

$9 \div 3 \neq 3 \div 9$

$\therefore$ division is not commutative.

Note that in the example 4 above, to prove that an operation is commutative as in (i) and (ii) the property had to be shown to be true for every pair of elements of the set, but to show that an operation is not commutative as in (iii) and (iv) it was sufficient to show that the property does not hold in the case of one pair of elements of the set. Thus, in proving any property for the elements of a set we have to show that it holds in respect of every element of it, but to disprove it, one single example where the property does not hold is sufficient. Such an example, which shows that the property does not hold is usually called a *counter example.*

2·5 (3) Associative property:

Let $*$ be a binary operation defined on a set S. $*$ combines any two elements of S to produce an element of S. What if we wish to combine more than two elements? Suppose a, b, c, are three elements of S and we wish to combine them under an operation $*$. We may do so in two ways. We may first combine a and b, and with the result combine c denoting the final result as $(a * b) * c$ .Or, we may first combine b and c and with he result, combine 'a' and get $a * (b * c)$. Is $(a*b)*c = a*(b*c)$?

Let us take a concrete example. Consider the set I of all integers and let the operation $*$ be the usual addition " + " defined in it. Take any three elements say 2, 7, 5 of I. We have,

$$(2*7) * 5 = (2 + 7) + 5 = 9 + 5 = 14$$

and $2* (7 * 5) = 2 + (7 + 5) = 2 + 12 = 14$

$$\therefore (2 * 7) * 5 = 2 * (7 * 5)$$

The result is the same, whatever way the computation is made. In fact for any three elements a, b, c of I,

$$(a + b) + c = a + (b + c).$$

Next, let the operation $*$ be the usual subraction "—" defined on the same set I. Take the same elements 2, 7, 5 of I. we have,

$$(2 * 7) * 5 = (2 - 7) - 5 = -5 - 5 = -10$$

and $2* (7 * 5) = 2 - (7 - 5) = 2 - 2 = 0$

So that $(2 * 7) * 5 \neq 2 * (7 * 5)$

Thus, in the case of subtraction in I, while combining three (or more) elements, the final result depends upon the order in which the elements are combined. For the operation of addition on the same set I however, the order of combination is immaterial. In such cases the operation is said to be *associative.*

Definition : If an operation $*$ defined on a set S is such that for any three elements a, b, c, of S, $(a * b) * c = a * (b * c)$, then $*$ is said to be **associative** in S.

Examples :

1. The operations of set union and intersection are associative. Because, for any three sets A, B, C,

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$\text{and } (A \cap B) \cap C = A \cap (B \cap C).$$

2. In the set N of natural numbers, addition and mutiplication are associative operations.

3. Let the operation $*$ be defined on N as
$a * b = a + 3b$.

Then, for any three elements a, b, c of N,

$(a * b) * c = (a + 3b) * c = a + 3b + 3c.$

And $a * (b * c) = a * (b + 3c) = a + 3(b + 3c) = a + 3b + 9c$

$\therefore (a * b) * c \neq a * (b * c)$

$\therefore *$ is not associative on N.

2·6 (4) Distributive Property :

We have seen in the previous chapter that for any three sets A, B, C,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$

and we call it the distributive property of '$\cap$' over '$\cup$'.

We know that in the set N of natural numbers,

$a. (b + c) = a. b + a. c.$ for any three elements a, b, c.

These are two examples where two binary operations are used on three elements of a system, and one operation is 'distributed' over the other.

Definition : If $*$ and $\triangle$ are two operations defined on a set S, and

$$a * (b \triangle c) = (a * b) \triangle (a * c)$$

for any three elements a, b, c of S, then $*$ is said to be **distributive** over $\triangle$.

Examples :

1. For any three sets A, B, C,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$\therefore$ the operation of set union is distributive over set intersection.

2. In the set R of reals, multiplication is distributive over addition. Because, $a.(b + c) = a.b + a.c$ for all a, b, c in R.

3. In the same set R of reals, addition is not distributive over multiplication.

Counter example of distributivity:

$$3 + (4 \cdot 5) \neq (3 + 4).\ (3 + 5)$$

2·7 (5) Identity

You have used the properties of the set of integers I many times. Did you notice any peculiar property of the element 'zero' in it with respect to addition? Observe that

$$3 + 0 = 0 + 3 = 3;$$
$$21 + 0 = 0 + 21 = 21;$$
$$-7 + 0 = 0 + (-7) = -7$$

and, in general, $a + 0 = 0 + a = a$ for any $a \in I$. Thus, whatever integer is added to 0, the result is the same number. Because of this property, 0 is called the "identity" element in I for addition. Is there any element in I which behaves similarly with respect to multiplication? What about 1?

We give the following definition.

Definition : If $*$ is an operation defined on a set A and e is an element in A such that

$$a * e = e * a = a$$

for every element $a \in A$, then e is called the **identity** element of A with respect to the operation $*$.

Illustrations :

(1) Let $A = \{0, 3, 7\}$ and $*$ be the operation of ordinary addition.

Then, $3 * 0 = 3 + 0 = 3$ and $0 * 3 = 0 + 3 = 3$

$\therefore 3 * 0 = 0 * 3 = 3$

Similarly, $7 * 0 = 0 * 7 = 7$

and $0 * 0 = 0$

So the element $0 \in A$ is such that if it is combined with any element of the set A, we get the same element.

$\therefore$ 0 is identity in A under $*$.

(2) Let $S = \left\{-2, 1, \frac{9}{4}\right\}$ and $*$ be the operation of ordinary multiplication.

Then, $-2 * 1 = -2 \cdot 1 = -2$ and $1 * -2 = 1 \cdot (-2) = -2$

$\therefore \quad -2 * 1 = 1 * (-2) = -2.$

Similarly, $1 * 1 = 1$

and $\frac{9}{4} * 1 = 1 * \frac{9}{4} = \frac{9}{4}$

So the element $1 \in S$ is such that when it is combined with any element of S, we get the same element.

$\therefore$ 1 is identity in S under $*$.

(3) Let $A = \{p, q, r\}$ and $*$ be an operation defined by the following table.

*	p	q	r
p	p	q	r
q	q	r	r
r	r	q	q

Then, $p * q = q * p = p$

$p * r = r * p = r$

$p * p = p$

Hence $p \in A$ is identity in A under $*$.

*2·8 Identity is unique :

If e is identity element in any set A with respect to the binary operation $*$, then e is unique; that is, no other element of A will play the same role as e does. For let, if possible, $e' \in A$ be another identity with respect to $*$

$\because$ e is an identity element, $\therefore$ for any $a \in A$,

$$a * e = e * a = a$$

$$\text{as } e' \in A \quad \therefore \quad e' * e = e * e' = e' \quad \ldots\ldots\ldots\ldots \quad (1)$$

Similarly, $\because$ e' is an identity element $\therefore$ for any $a \in A$,

$$a * e' = e' * a = a$$

$$\text{as } e \in A \quad \therefore \quad e * e' = e' * e = e \quad \ldots\ldots\ldots\ldots \quad (2)$$

from (1) and (2) $e = e'$

Thus, whenever identity exists, it is unique.

2·9 Inverse :

Consider the set I of all integers and the operation '+' of ordinary addition defined in it.

We know that the identity element is 0. Observe that the pairs 3,—3; 7, —7; 13, —13 are such that the addition of the elements in each pair is 0. In fact, if 'x' is any number in I, then we can find another number $-x$ in I such that

$$x + (-x) = (-x) + x = 0$$

The elements forming such pairs as x, $-x$ are called the inverse of each other with respect to '+' in I.

In general, we define "inverse" as follows :

Definition : Let $*$ be an operation defined on a set A, and let e be the identity in A with respect to $*$. If to an element $a \in A$, there exists an element $b \in A$ such that

$$a * b = b * a = e$$

then a and b are called the **inverse** of each other with respect to $*$ in A.

Illustrations :

(1) Let $A = \{-1, 0, 1, 2,\}$ and $*$ be ordinary addition.

$\because *$ is addition $\therefore$ 0 is the identity.

Now, $-1 * 1 = 1 * (-1) = 0$ $\therefore$ -1 and 1 are inverse of each other.

2 has no inverse.

$0 * 0 = 0$ $\therefore$ 0 is its own inverse.

(2) Let $A = \left\{1, 2, \frac{1}{2}\right\}$ and $*$ be ordinary multiplication.

$\because *$ is multiplication $\therefore$ 1 is the identity. Also, $1 * 1 = 1$ $\therefore$ 1 is its own inverse and, $2 * \frac{1}{2} = \frac{1}{2} * 2 = 1$ $\therefore$ 2 and $\frac{1}{2}$ are inverse of each other.

(3) Let $S = \{a, b, c\}$ and $*$ be the operation defined by the following table

*	a	b	c
a	a	b	c
b	b	a	a
c	c	b	a

As is evident from the table, 'a' is the identity element.

Also, $b * b = a$ $\therefore$ b is its own inverse.

and $c * c = a$ $\therefore$ c is its own inverse.

As usual, the identity "a" is its own inverse.

2·20 Properties of N, I and R

1. The system of natural numbers N.

   (i) It is closed with respect to addition and multiplication.
   For any two elements a, b of N,
   $a + b \in N$ and $a \cdot b \in N$.

   (ii) Addition and multiplication are commutative operations in N.

   For any two elements a, b of N,

   $a + b = b + a$ and $a \cdot b = b \cdot a$

   (iii) Subtraction and division are not defined in it.
   Consider the elements $2 \in N$ and $3 \in N$.
   $2 - 3 = -1$ but -1 is not an element of N.

Similarly, consider the elements $4 \in N$ and $7 \in N$

$$4 \div 7 = \frac{4}{7} \text{ but } \frac{4}{7} \text{ is not an element of N.}$$

   (iv) Addition and multiplication are associative operations in N.

   For any three elements a, b, c of N,

   $$a + (b + c) = (a + b) + c \text{ and}$$
   $$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

   (v) It does not possess any identity element with respect to addition.

   (vi) The element $1 \in N$ is the identity with respect to multiplication

   For any element $a \in N$,
   $a \cdot 1 = 1 \cdot a = a$.

   (vii) Inverses do not exist with respect to addition or multiplication.

2. The system of integers I

   (i) It is closed with respect to addition, subtraction and multiplication.

(ii) Addition and multiplication are commutative operations in I.

(iii) Subtraction is not commutative in I.
Counter-example of commutativity : $3 - 5 \neq 5-3$.

(iv) Addition and multiplication are associative in I.
For any three elements a, b, c of I,
$a + (b + c) = (a + b) + c$ and
$a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

(v) Subtraction is not associative in I.
Counter-example of associativity: $5 -(4-3) \neq (5-4) -3$.

(vi) The element $0 \in I$ is the identity with respect to addition and the element $1 \in I$ is the identity with respect to multiplication.

(vii) There is no identity element in I with respect to subtraction.

(viii) Inverses exist with respect to addition.
For any element $a \in I$, there exists an element $(-a) \in I$ such that $a + (-a) = 0$.

For example, consider any integer, say 31 of I. Corresponding to it, there exists an integer —31 of I such that $31 + (-31) = 0$.

(ix) Inverses do not exist with respect to subtraction or multiplication.

3. The system of real numbers R.

(i) It is closed with respect to addition, subtraction, multiplication and division (except by 0).

(ii) Addition and multiplication are commutative in R.

(iii) Subtraction and division are defined in it, but neither of them is commutative in R. Division by 0 is undefined.

(iv) Addition and multiplication are associative in R, but neither subtraction nor division is associative in it.

(v) The elements $0 \in R$ and $1 \in R$ are the identities with respect to addition and miltiplication respcetively.

(vi) There is no identity with respect to subtraction or division.

(vii) The inverses exist with respect to addition and multiplication but not with respect to subtraction or division. e. g. with respect to addition, the inverse of 2 is -2 as

$2 + (-2) = 0$; inverse of $-\frac{3}{5}$ is $\frac{}{5}$ as

$$\left(-\frac{3}{5}\right) + \frac{3}{5} = 0$$

In general, inverse of any element $a \in R$ is $(-a) \in R$ (with respect to addition)

for, $a + (-a) = 0$.

What is the inverse of 0 under addition?

With respect to multiplication, the inverse of $-\frac{7}{9}$ is $-\frac{9}{7}$ for

$$\left(-\frac{7}{9}\right)\left(-\frac{9}{7}\right) = 1.$$

In general, the inverse of any element $a \in R$ is $\frac{1}{a} \in R$ $(a \neq o)$, for,

$$a\left(\frac{1}{a}\right) = 1$$

The inverse of 1 under multiplication is 1 itself.

Note that 0 has no inverse under multiplication, for $\frac{1}{0}$ is undefined.

* 2·21 Group

Any set S with a binary operation* defined in it is called a "group" provided it possesses the following properties.

(1) It is closed under *.

That is, for every pair of elements a, b, $a \in S$ and $b \in S$ implies $a*b \in S$.

(2) $*$ is associative in S.

That is, whatever elements a, b, c of S we may choose,

$$(a * b) * c = a * (b * c).$$

(3) It possesses an identity under $*$ which is unique.

That is, there exists an element $e \in S$ such that for every $a \in S$,

$$a * e = e * a = a.$$

(4) Each element in S possesses an inverse under $*$.

That is, for every $a \in S$, there exists an element $x \in S$ such that

$$a * x = x * a = e.$$

Note that

(a) I is a group under addition, but not multiplication. (Why not?)

(b) R is a group under addition but not multiplication. (Why not?)

(c) N is not a group under addition or multiplication (Why?)

EXERCISE 1

1. Find if each of the following sets is closed under addition. If not, state why.

(i) $\{0, 1, 2\}$; (ii) $\{-1, 0, 1, 2\}$; (iii) $\{1, 2, 3, 4\}$; (iv) $\{0\}$;
(v) $\{2, 4, 6, 8, \ldots\ldots\ldots\}$; (vi) $\{1, 2, 4, 6, 8 \ldots\ldots\ldots\}$;
(vii) $\{-1, 0, 1, 5\}$; (viii) $\{0, 4, 8, 12, 16, 20, \ldots\ldots\ldots\}$;
(ix) $\{1, 3, 5, 7 \ldots\ldots\ldots\}$;

2. Find if each of the following sets is closed under multiplication. If not, state why.

(i) $\{1, 0\}$; (ii) $\{-1, 1\}$; (iii) $\{2, 3, 9, 5\}$; (iv) $\{1, 2, 4, 6, 8 \ldots\ldots\}$;
(v) $\{2, 4, 6, 8 \ldots\}$; (vi) $\left\{6, \frac{1}{6}, 1, 9, \frac{1}{9}\right\}$; (vii) $\left\{1, 3, \frac{1}{3}, 2, \frac{1}{2}\right\}$;

(viii) $\{1, 3, 5, 7, 9 \ldots\ldots\}$; (ix) $\{-5, 0, 5\}$; (x) $\left\{\frac{1}{2}, 1, 0, -\frac{1}{2}\right\}$.

3. Find the identity and inverses of the elements of the following sets with respect to usual addition.

(i) $A = \{-1, 0, 1, 2\}$; (ii) $B = \{2, 3, -3, -2\}$;
(iii) $D = \{-4, -5, -7, 0\}$; (iv) $E = \{p, q, r, o, -r, -q, -p\}$.

4. Let the operation $*$ be defined on the set $S = \{a, b, c\}$ by the following table.

*	a	b	c
a	a	b	c
b	b	a	c
c	c	b	b

Find if $*$ is

(i) commutative;

(ii) possesses an identity; also find

(iii) the inverses of elements of S under $*$.

In the set N of natural numbers, find if the following operations are commutative. Give a counter-example in each case where your answer is "no".

5. The operation $*$ defined as : $a * b = 5a + 7b$

6. The operation $\otimes$ defined as $a \otimes b = a^2 + b^2$.

7. The operation $\ominus$ defined as $a \ominus b = (3a)(4b)$.

8. The operation $\bullet$ defined as $a \bullet b = \frac{a+b}{2}$.

9. The operation $\times$ defined as $a \times b = \frac{6a + 5b}{7}$.

10. In the set R of all real numbers, verify whether the binary operation $*$ defined as under is in each case commutative.

(i) $a * b = a - b + ab$; (ii) $a * b = a + b + ab$;
(iii) $a * b = (3a - 4b)^2$; (iv) $a * b = (a-b)^2$; (v) $a * b = a + b - ab$.

In the set I of all integers, find if the operation $*$ defined as under is associative. Give a counter-example in each case where your answer is "no".

11. $a * b = \frac{a + b}{2}$

12. $a * b = (5a)\ b$

13. $a * b = 2a + b$

14. In the set I of all integers, operation $*$ is defined as $a * b = a^2 + ab + b^2$. Find if $*$ possesses (i) closure, (ii) commutativity,

15. The operation ⟠ on the set $A = \{\omega, \theta\}$ is defined by the following table.

⟠	ω	θ
ω	ω	θ
θ	θ	ω

Find if ⟠ possesses (i) closure;

(ii) commutativity.

16. In the previous example, does ⟠ possess an identity? If so find it.

17. In ex. 15, prove that each element of A is its own inverse.

18. The operations $\otimes$ and $*$ on the set $A = \{\theta, \psi, \mu\}$ are defined by the following tables :

⊗	θ	ψ	μ
θ	μ	θ	ψ
ψ	ψ	μ	θ
μ	θ	ψ	μ

*	θ	ψ	μ
θ	θ	ψ	μ
ψ	ψ	μ	θ
μ	μ	θ	ψ

Find if these operations on the set A possess (i) closure; (ii) commutativity. Justify your answer.

19. In the previous example, does the set A possess an identity with respect to (i) the operation ⊗, (ii) the operation * ?

20. The operation * on the set S = {α, β, γ, δ} is defined by the following table :

*	α	β	γ	δ
α	α	β	γ	δ
β	β	γ	δ	α
γ	γ	δ	α	β
δ	δ	α	β	γ

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Find if the operation * on the set S is (i) closed, (ii) commutative, (iii) associative.

21. In the previous example, does * possess an identity? If yes, then find the inverse of each element of S with respect to * .

22. In the set N of natural numbers, let $*$ be the binary operation defined as "take the maximum of". That is, $2 * 5 = 5$, $13 * 11 = 13$ etc. Compare $6 * 9$ with $9 * 6$. Is $6 * 9 = 9 * 6$?

In general if $a \in N$ and $b \in N$, is $a * b = b * a$? i. e., is $*$ commutative? Is it associative? If not, give counter-example.

23. In the previous example, does N possess an identity with respect to $*$?

24. Let the same operation $*$ as defined in ex. 22 be defined in the set $A = \{0, 1, 2, 3 \ldots\ldots\ldots\ldots\}$. Does A possess an identity with respect to $*$?

25. In the set N of natural numbers, let the operation $\otimes$ be defined as "take the minimum of". Is $\otimes$ commutative? Is it associative?

26. In the previous example, does N possess an identity with respect to $\otimes$?

27. Prove the cancellation laws in the system of real numbers;

(i) If $a + b = a + c$, then $b = c$;
(ii) If $ab = ac$, then $b = c$. $(a \neq 0.)$

[Proof (i) $a + b = a + c$ Given

$$\begin{aligned} \therefore\ & (-a) + (a + b) = (-a) + (a + c) & \\ \therefore\ & (-a + a) + b = (-a + a) + c & \text{associativity} \\ \therefore\ & 0 + b = 0 + c & \text{inverse} \\ \therefore\ & b = c & \text{identity]} \end{aligned}$$

*28. Prove that (i) $a \times 0 = 0$ (ii) $(-a)(-b) = ab$
(iii) $(a)(-b) = (-a)(b) = -ab$

[proof (ii)

$$\begin{aligned} (-a)(-b) &= (-a)(-b) + 0 & \text{identity} \\ &= (-a)(-b) + b \times o & \text{from (i)} \\ &= (-a)(-b) + b\,[(-a) + a\,] & \text{inverse} \end{aligned}$$

$$\begin{aligned}
&= (-a)(-b) + b(-a) + ba && \text{distributivity}\\
&= (-a)(-b) + (-a)\,b + ab && \text{commutativity}\\
&= (-a)[-b + b] + ab && \text{distributivity}\\
&= (-a) \times 0 + ab && \text{identity}\\
&= 0 + ab && \text{from (i)}\\
&= ab && \text{identity}
\end{aligned}$$

29. If the operation $*$ on the set $\{p, q, r, s\}$ is commutative, fill the blank squares in the following table.

*	p	q	r	s
p	p	r		s
q		s	q	p
r	q	s	r	
s	p		q	p

30. Let $A = \{a, b, c, d\}$ and $*$ be a binary operation defined on it. Given 'a' is the identity element with respect to $*$, fill the blanks in the following table.

*	a	b	c	d
a				
b		c	d	a
c		d	a	b
d		a	b	c

ANSWERS

1. (i) No; (ii) No; (iii) No; (iv) Yes; (v) Yes; (vi) No; (vii) No; (viii) Yes; (ix) No.

2. (i) Yes; (ii) Yes; (iii) No; (iv) Yes; (v) Yes; (vi) No; (vii) No; (viii) Yes; (ix) No; (x) No.

3. (i) Identity 0. 1, −1 are inverse of each other. 2 has no inverse.
   (ii) No identity and therefore no question of inverse.
   (iii) Identity 0. Except 0 none of the elements possesses inverse.
   (iv) Identity 0. Inverse of o, p, q, r, are o, − p, −q, −r respectively.

4. (i) Not commutative $c * b \neq b * c$.
   (ii) Identity is a.
   (iii) Inverses of a and b are a and b respectively. c has no inverse.

5. No; 6. Yes; 7. Yes; 8. Yes; 9. No.

10. (i) No; (ii) Yes; (iii) No; (iv) Yes; (v) Yes.

11. No; 12. Yes; 13. No; 14. (i) Yes; (ii) Yes.

15. (i) Yes; (ii) Yes; 16. Yes, ω is the identity.

18. (i) A is closed under both the operations.
    (ii) Commutative under $*$ but not under $\otimes$

19. (i) No; (ii) Yes; θ is the identity.

20. (i) Yes; (ii) Yes; (iii) Yes.

21. Yes. The identity is α. The inverses of $\alpha, \beta, \Upsilon, \delta$ are $\alpha, \delta, \Upsilon, \beta$ respectively.

22. Yes, Yes. 23. Yes. 1 is the identity.

24. Yes. 0 is the identity.

25. Yes, Yes. 26. No. 29. q, r, q, p rowwise.

30. The first row and the first column should each be written as a, b, c, d.

2·22 Ordered pairs

We know that while listing the elements in a set, the order is considered to be of no importance. So, in a set of a pair of elements, the position that any of them occupies within the braces is unimportant. Thus {5, 9} is the same as {9, 5} ; {eat, run} is the same as {run, eat}, and {husband, wife} is the same as {wife, husband}. There are instances, however, when we wish to attach specific meaning to the position of an element of a given pair. For example, in a table tennis singles, the first number of the score shows the points gained by the player making the service, and the second, those gained by the other player. So, the score 5—9 does not convey the same meaning as the score 9—5. If Ramesh and Mahesh are playing, Ramesh servicing and Mahesh receiving, then 5—9 will mean Ramesh has gained 5 points and Mahesh has gained 9 points, whereas 9—5 will have a completely different meaning. It will mean Mahesh has 5 points and it is Ramesh who has gained 9 points. Hence in this case, it is very important as to whether a number is written in the first position or the second.

Similarly, in the sequence of works, run—eat which means run then eat, is definitely different from and wiser than eat—run which means eat and then run. Again, on a nameplate if you find M. A., Ph.D. it is quite all right; but can you ever think of it as being written Ph. D. M.A.? Take one more example. In the process of dressing, one puts on the shirt before the tie. So, in the sequence of dressing up, shirt—tie is meaningful but not tie—shirt.

The examples discussed above clearly show that there are matters in which a pair of elements should be written with some specific meaning assigned to the order in which they are written. If two elements x and y are to be written in a specific order, so that x is to occupy the first position and y the second, and the place value of the elements has a meaning, then we generally use parenthesis writing it as (x, y) and call it an ordered pair.

An ordered pair, therefore, is a pair of objects, one of which is assigned the first position and is called the first component while the other is assigned the second position and is called the second component.

The position of an element in an ordered pair, once fixed, must continue to have the same meaning throughout the discussion. In the example of the table tennis score given above, the first number shows the number of points obtained by the servicing player. This understanding, attached with the first number of the score, remains unaltered throughout the game. Similarly for the second number in the score.

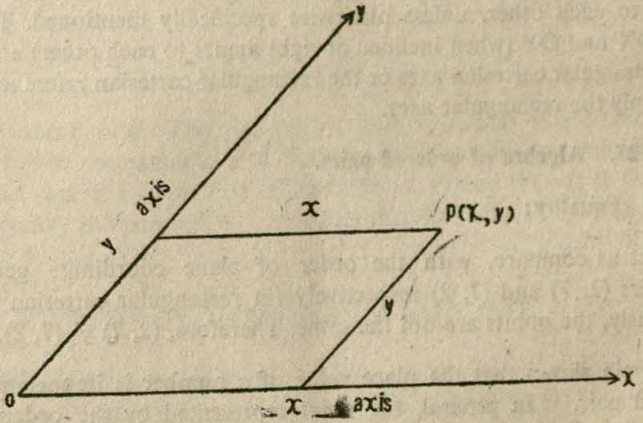
In general, if (a, b) is to be an ordered pair, 'a' must always carry the same meaning which is assigned to the first component of the ordered pair under the particular discussion, and 'b' must always fulfil the condition laid down for the second component of the same.

It is possible that in a particular discussion with a certain order, both the components of an ordered pair may be the same number. Going back to the example of the table tennis score again, we know that a situation may arise when both the players have got 5 points; and then the score will be written as 5-5 that is, (5, 5). So, the components in an ordered pair need not necessarily be different. The pairs (9, 9), $(-2, -2)$, (x, x), (α, α) are therefore valid pairs under, of course, the reference of order of each. However, even if the components are the same numbers, an ordered pair will always be written as a pair and not a single element. For example, the ordered pair (9, 9) will never be written as (9), although the set {9, 9} is the same as {9}.

2·23. The notion of ordered pairs has a very wide use, one among which is the construction of numbers. A rational number, say $\frac{3}{4}$, can be thought of as an ordered pair (3, 4) with the order that the first component is the integer in the numerator and the second component (not equal to zero), the integer in the denominator. With this order, therefore, every rational number $\frac{x}{y}\left(y \neq 0\right)$ is an ordered pair (x,y). Note that the pair (3,4) which means $\frac{3}{4}$ is not the same as the pair (4, 3) which stands for $\frac{4}{3}$. In general, (x, y) is not the same as (y, x) except when $x = y$.

The utility of this interpretation of a fraction in terms of an ordered pair is outside the scope of the present discussion, but it has very important and powerful use in the construction of numbers.

Another use of this notion is made in plane coordinate geometry. In this, a fixed point, say O, is called the origin and two fixed lines OX and OY passing through O are called the axis of x and the axis of y respectively.



Any point P in the plane is then represented by the ordered pair (x, y) where x is the distance of P from y axis measured parallel to OX, and y is the distance of P from x axis measured parallel to OY. With this order understood, any ordered pair (3, 9) in this geometry represents a fixed point which is to be located by measuring 3 units along the x axis and then 9 units parallel to the y axis. Thus any ordered pair can be represented by a point in the plane and any point can be represented by an ordered pair in plane coordinate geometry.

Example:

Locate the points (6, 7), (5, 5), (- 4,9), (0, 4), (10, −7), (5, 0), (−2, −8). (0, −9), (−9, 0), (−3, −3), $\left(2\frac{1}{2}, -\frac{3}{2}\right)$ with the order mentioned above for plane coordinate geometry, taking OX and OY perpendicular to each other. What is the ordered pair representing the origin O?

2·24. This procedure, namely that of representing a point by an ordered pair and conversely, is very useful and important and a useful branch of mathematics called "coordinate geometry" is based on it. We shall be making use of this concept in later chapters. For The sake of convenience, we shall hereafter always take the axes to be inclined at right angles to each other, unless otherwise specifically mentioned. The two axes OX and OY (when inclined at right angles to each other) are called the rectangular cartesian axes or the rectangular cartesian reference frame or simply the rectangular axes.

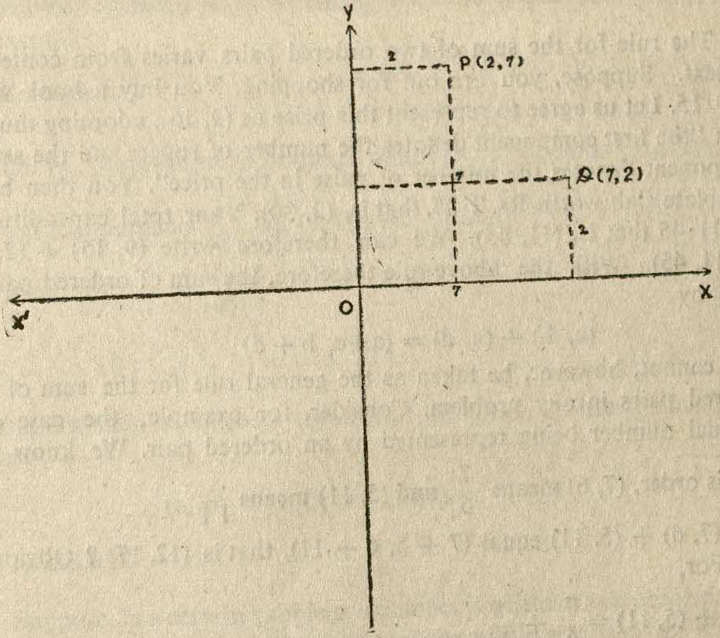
2·25. Algebra of ordered pairs.

1. Equality:

Let us compare, with the order of plane coordinate geometry, the pairs (2, 7) and (7, 2) respectively (in rectangular cartesian frame). Obviously, the points are not the same. Therefore, $(2, 7) \neq (7, 2)$.

This again shows that the place value of a number is important in an ordered pair. In general, the point represented by the ordered pair (x, y) will not be the same as that represented by (y, x) except when $x = y$.

In the example of rational numbers being represented by ordered pairs, if the number say $\frac{5}{6}$, which is (5, 6) is denoted by (x, y) then clearly,



x = 5 and y = 6. That is, (x, y) = (5, 6) means x = 5 and y = 6. Similarly in the example of a point in the plane being expressed as an ordered pair, if a point P is taken to be (x, y) and then it is found that P is actually the point (9, 11), we know that the value of x is 9 and that of y is 11. So,

$$(x, y) = (9, 11) \text{ implies } x = 9 \text{ and } y = 11.$$

In general, we make the following definition for the equality of two ordered pairs :

Definition

(i) (a, b) = (c, d) if and only if a = c and b = d.

(ii) (a, b) = (b, a) if and only if a = b.

2. Sum of ordered pairs

The rule for the sum of two ordered pairs varies from context to context. Suppose, you are out for shopping. You buy a book worth Rs 9·15. Let us agree to represent this price as (9, 15), adopting thus the rule: "the first component denotes the number of rupees and the second component denotes the number of paise in the price". You then buy a chocolate slab worth Rs. 2·30, that is, (2, 30). Your total expenditure is Rs. 11·45 that is, (11, 45). We can therefore write (9, 15) + (2, 30) = (11, 45). With the above rule therefore, the sum of ordered pairs is given by

$$(a, b) + (c, d) = (a+c, b+d)$$

This cannot, however, be taken as the general rule for the sum of two ordered pairs in any problem. Consider, for example, the case of a rational number being represented by an ordered pair. We know that in this order, (7, 6) means $\frac{7}{6}$, and (5, 11) means $\frac{}{11}$

Will (7, 6) + (5, 11) equal (7 + 5, 6 + 11), that is, (12, 17) ? Obviously not. For,

$$\begin{aligned}(7, 6) + (5, 11) &= \frac{7}{6} + \frac{5}{11}\\ &= \frac{7\cdot 11 + 5\cdot 6}{6\cdot 11}\\ &= \frac{107}{66}\\ &= (107, 66)\end{aligned}$$

So, for rational numbers,

$$(a, b) + (c, d) \neq (a + c, b + d)$$

The rule for summation may, therefore, be made to depend upon the meaning we assign to the components.

3. Multiplication of ordered pairs

Just as the rule for the addition of ordered pairs depends on the meaning assigned to its compoents, and varies from context to context, so does the rule for multiplication. That the rule for multiplication must be specified separately in each context would be evident from the following examples.

Example 1.

Let (x, y) represent the rational number $\frac{x}{y}$, ($y \neq 0$). Consider any two rational numbers $\frac{3}{10}$ i. e., (3, 10) and $\frac{5}{7}$ i. e., (5, 7).

Now, $\frac{3}{10} \times \frac{5}{7} = \frac{15}{70}$

$\therefore$ (3, 10) × (5, 7) = (15, 70).

So, in the system of rational numbers, the rule for multiplication is given by

$$(a, b) \times (c, d) = (ac, bd)$$

Example 2.

Suppose, in a certain problem, a number is written as an ordered pair, the first component of which represents the whole part and the second component represents the fractional part. For example, $5\frac{1}{2}$ is written as $\left(5, \frac{1}{2}\right)$, $\frac{7}{3}$ is written as $\left(2, \frac{1}{3}\right)$ and so on. The rule (a, b) × (c, d) = (ac, bd) will not be obeyed by the ordered pairs formed in this context.

For, consider any two ordered pairs, say, $\left(3, \frac{1}{4}\right)$ which is $3\frac{1}{4}$ and $\left(9, \frac{1}{2}\right)$ which is $9\frac{1}{2}$.

Then, $\left(3, \frac{1}{4}\right) \times \left(9, \frac{1}{2}\right) = \left(3\frac{1}{4}\right) \times \left(9\frac{1}{2}\right)$

$$= \frac{13}{4} \times \left(\frac{19}{2}\right)$$
$$= \frac{247}{8}$$
$$= 30\frac{7}{8}$$
$$= \left(30, \frac{7}{8}\right)$$

So that,

$$\left(3, \frac{1}{4}\right) \times \left(9, \frac{1}{2}\right) \neq \left(3 \times 9, \frac{1}{4} \times \frac{1}{2}\right)$$

Exercise:

1. If $(x, 3) = (9, 3)$ find x.
2. If $(5, m) = (5, -10)$ find m.
3. If $(6, 7) = (-y, 7)$ find y.
4. If $(-3, 1) = (5r, 1)$ find r.
5. If $(x + y, 1) = (3, x - y)$ find x and y.
6. Find the values of x and y if $(y-2, 2x + 1)$ equals $(x-1, y+2)$
7. Let addition in ordered pairs be defined as :
   $(a, b) + (c, d) = (a + d, b + c)$.

Compute the following :

(i) $(2, 3) + (-6, -8)$

(ii) $\left(\frac{3}{2}, -7\right) + \left(17, \frac{5}{2}\right)$

(iii) $\left(-\frac{2}{3}, \frac{9}{4}\right) + \left(-\frac{3}{2}, \frac{11}{3}\right)$

(iv) $(59, 0) + \left(7, -\frac{1}{2}\right)$

(v) $\left(\frac{12}{5}, -k\right) + \left(1, -\frac{12}{5}\right)$

8. Let multiplication in ordered pairs be defined as :

$$(x, y) \times (s, t) = (xt, ys)$$

Compute the following:

(i) $\left(6, 7\frac{1}{2}\right) \times \left(\frac{1}{2}, 4\right)$

(ii) $\left(-3\frac{1}{3}, 0\right) \times \left(0, -7\right)$

(iii) $\left(0, \frac{125}{9}\right) \times \left(0, -\frac{69}{2}\right)$

(iv) $\left(-\frac{3}{2}, -\frac{5}{6}\right) \times \left(1, -8\right)$

(v) $\left(-11, 0\right) \times \left(p, a\right)$

9. Let multiplication in ordered pairs be defined as :

$$(a, b) \times (c, d) = (ac - bd, ad + bc)$$

Compute the following :

(i) $(1, 1) \times (5, 3)$

(ii) $(-2, 2) \times \left(-\frac{1}{2}, \frac{3}{2}\right)$

(iii) $(\sqrt{3}, 7) \times (-1, \sqrt{2})$

(iv) $\left(\frac{4}{3}, \frac{1}{5}\right) \times \left(\frac{6}{5}, \frac{3}{2}\right)$

(v) $(a, 11) \times (2, b)$

10. If the addition in ordered pairs is defined as:

$$(a, b) + (c, d) = (ac - d, bd - c),$$

find the values of m and k, given that

$$(m, k) + (6, -5) = (11, -16).$$

Answers

(1) 9 , (2) -10 , (3) -6 , (4) $-\frac{3}{5}$,

(5) $2, 1$, (6) $2, 3$,

(7) (i) $(-6, -3)$, (ii) $(4, 10)$, (iii) $\left(3, \frac{3}{4}\right)$,

(iv) $\left(58\frac{1}{2}, 7\right)$, (v) $\left(0, 1-k\right)$,

(8) (i) $\left(24, \frac{15}{4}\right)$, (ii) $\left(\frac{70}{3}, 0\right)$, (iii) $(0, 0)$,

(iv) $\left(12, -\frac{5}{6}\right)$, (v) $\left(-11a, 0\right)$

(9) (i) $\left(2, 8\right)$, (ii) $\left(-2, -4\right)$,

(iii) $\left(-\sqrt{3} - 7\sqrt{2}, \sqrt{6} - 7\right)$,

(iv) $\left(\frac{13}{10}, \frac{56}{25}\right)$, (v) $\left(2a - 11b, ab + 22\right)$.

(10) $1, 2$.

2·26 Matrices :

In the middle of the last century, many eminent mathematicians notably Cayley and Sylvester, developed a way of arranging numbers in ordered arrays and called this arrangement a "matrix". The matrix algebra was developed steadily since then and today it is one of the most useful, important and also one of the most fascinating branches of Mathematics. Besides the applications of matrices in Statistics, Economics, Complex algebra and other theoretical structures, it has very wide applications in almost every branch of technology and engineering. In the study of atomic physics matrices are indispensable. The algebra of matrices is in fact the language of atomic physics. In the study of quantum theory, the theory of relativity, linear programming, electrical networks and computers, matrices are used now in every branch of scientific progress.

A "matrix" is an array of real numbers arranged in rows and columns.

For example, each of the arrays

$$\begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 6 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 5 \\ -2 & 8 & 9 \\ 6 & \frac{1}{2} & 3 \end{bmatrix}$$

is a matrix. In each of the first and second, there are two rows and two columns. They are called 2 × 2 (read two by two) matrices. The third has three rows and three columns and is called a 3 × 3 matrix.

A matrix containing only one column is called a column matrix.

e. g. $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

is each a column matrix.

A matrix containing only one row is called a row matrix.

e. g. [a, b, c, d], [0, 0], [−1, 2, 9]

is each a row matrix.

A matrix may have any number of rows and columns. In our discussion, however, the number of rows and columns will never go beyond 2 each.

2·27. Algebra of Matrices :

Having defined an ordered arrangement of real numbers, we next proceed to consider the set of such arrangements (each matrix being considered as an entity) and define some operations in it, so as to be able to pursue the subject further.

Definition 1. Equality of matrices:

Two matrices A and B are said to be equal, provided

(i) A and B are of the same size, that is, they have the same number of rows and columns.

(ii) Each component of A is equal to the corresponding component of B.

For example, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

if and only if $a = p$, $b = q$, $c = r$, and $d = s$.

2·28. Matrix addition :

The addition of matrices is defined to be a binary operation such that the sum of two matrices of equal order is another matrix of the same order in which each component is the sum of the corresponding components of the two matrices.

For example, let $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

and $B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$

Then $A + B = \begin{bmatrix} x + \alpha & y + \beta \\ z + \gamma & w + \delta \end{bmatrix}$

Note that in order that two matrices may be added, it is necessary that they should be of the same order.

2·29. The zero matrix or the null matrix :

Consider the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. We find that if A be a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c, d are any real numbers, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Thus the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is such that if it is added to any matrix A of the same order, the result is the matrix A itself.

The matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is called the null matrix or zero matrix and plays the same role as 0 does in real number system (with respect to real addition). It is the identity element with respect to matrix addition in the set of 2 × 2 matrices. It is sometimes denoted by 0. To avoid confusion with the zero of the real number system, we shall denote the null matrix by [0].

2·30. Properties of matrix addition:

1. *Matrix addition is commutative.*

Consider any two matrices $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$

and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

$$\therefore A + B = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix} \ldots\ldots \quad \text{(i)}$$

Since $a_1, b_1, a_2, b_2 \ldots\ldots$ are real numbers, $a_1 + b_1 = b_1 + a_1$ and so on.

$$\therefore \text{ from (i), } A + B = \begin{bmatrix} b_1 + a_1 & b_2 + a_2 \\ b_3 + a_3 & b_4 + a_4 \end{bmatrix}$$

$$= \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} + \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = B + A.$$

Hence addition in matrices is commutative.

2. *Matrix addition is associative* :

It is left for the students to prove that if A, B, C be any three matrices, $A + (B + C) = (A + B) + C$ where $+$ is the matrix addition.

3. *Existence of identity* :

We have seen in art. 2·29 that there exists a matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ denoted by $[\,0\,]$ such that for any 2×2 matrix A,

$$A + [0] = A.$$

$\therefore$ by commutativity, there exists a matrix $[\,0\,]$ such that, for every 2×2 matrix A,

$$A + [0] = [0] + A = A.$$

The matrix $[\,0\,]$ is, therefore, the identity with respect to addition in matrices.

4. *Existence of inverse :*

Let A be a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c, d are any real numbers. Then there exists a matrix $\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$ such

that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ……………… (i)

that is, to each matrix A, there exists another matrix such that when A is added to this matrix, the result is the identity element. Hence each 2 × 2 matrix possesses its inverse under addition.

We now leave one question open to students. Is the set of all 2 × 2 matrices a Group?

2·31. Scalar multiplication of a matrix :

Just as in the system of real numbers,

2a is a + a,

3a is a + a + a, and so on, and since the matrix addition depends directly on the addition in real number system, we have,

$$2\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}$$

Similarly, $3\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$

In general,

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

where k is any real number. The number k is called a scalar and the equality above shows that multiplying a matrix by a scalar means multiplying each element of the matrix by that scalar.

Examples : Find (i) $5\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$, (ii) $(-3)\begin{bmatrix} \frac{1}{2} & -\frac{1}{6} \\ -\frac{2}{3} & 4 \end{bmatrix}$

(iii) $-\begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}$

Solutions :

(i) $$5\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 5\times2 & 5\times0 \\ 5\times-1 & 5\times3 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ -5 & 15 \end{bmatrix}$$

(ii) $$-3\begin{bmatrix} \frac{1}{2} & -\frac{1}{6} \\ -\frac{2}{3} & 4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ 2 & -12 \end{bmatrix}$$

(iii) $$-\begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} = (-1)\begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}$$

2·32. Matrix Multiplication :

Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

be any two matrices. Then the matrix multiplication (denoted by × or ·) is defined to be a binary operation such that

$$A \cdot B = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \cdot \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$$
$$= \begin{bmatrix} a_1 b_1 + a_2 b_3 & a_1 b_2 + a_2 b_4 \\ a_3 b_1 + a_4 b_3 & a_3 b_2 + a_4 b_4 \end{bmatrix}$$

The rule being : Each element of the first row of matrix A is multiplied to the corresponding element of the first column of matrix B. The products are added and the result constitutes the first element of the resulting matrix. The elements of the first row of A are then multiplied to the corresponding elements of the second column of B, the products are added and the result constitutes the second element of the 1st row of the resulting matrix. Similarly with the second row of the first matrix.

$$\text{Thus, } \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} \times \begin{bmatrix} 6 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2\cdot 6+3\cdot 2 & 2\cdot 1+3\cdot 4 \\ 1\cdot 6+5\cdot 2 & 1\cdot 1+5\cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 14 \\ 16 & 21 \end{bmatrix}$$

2·33 The Unit matrix :

Consider the matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If A be any matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$A.\, I = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a\cdot 1+b\cdot 0 & a\cdot 0+b\cdot 1 \\ c\cdot 1+d\cdot 0 & c\cdot 0+d\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\text{and } I\cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot a+0\cdot c & 1\cdot b+0\cdot d \\ 0\cdot a+1\cdot c & 0\cdot b+1\cdot d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Thus there exists a matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ such that for any 2 × 2 matrix A, AI = IA = A.

This matrix is the identity element with respect to matrix multiplication; it is denoted by I and is called the unit matrix.

2·34. Some Properties of matrix multiplication :

1. *Matrix multiplication is non-commutative.*

Let A $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and B $= \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ be any two 2×2 matrices.

$$\text{Then, } AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

$$= \begin{bmatrix} ax+bz & ay+bt \\ cx+dz & cy+dt \end{bmatrix}$$

$$\text{and } BA = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} xa+yc & xb+yd \\ za+tc & zb+td \end{bmatrix}$$

so that $AB \neq BA$.

Thus matrix multiplication is not commutative. Note that though in general, commutativity does not hold for multiplication in matrices, every matrix commutes with I.

2. *The product of two non-zero matrices, may be zero.*

Let A $= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and B $= \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$

Then, $A \neq [0]$, $B \neq [0]$.

$$\text{Consider } A \cdot B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

i. e. $A \cdot B = [0]$.

So, if the product of two matrices is zero, it does not necessarily mean that either of the matrices has to be the null matrix.

Can you find some other pairs of non-zero matrices such that their product is zero?

The two properties of matrix multiplication stated above stand in contrast with the properties of multiplication in reals, in which (i) multiplication is commutative i. e., for all real numbers x, y, $x \cdot y = y \cdot x$. and (ii) if $x \cdot y = 0$ then either $x = 0$ or $y = 0$.

Thus of what we know of matrix algebra, it is different from the algebra of reals in the above two respects.

3. *Associative law of multiplication:*

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C$$

4. *Distributive laws:*

(i) Left distributive law

$$A \cdot (B + C) = A \cdot B + A \cdot C.$$

(ii) Right distributive law.

$$B + C) \cdot A = B \cdot A + C \cdot A.$$

It is left for the students to prove the above laws.

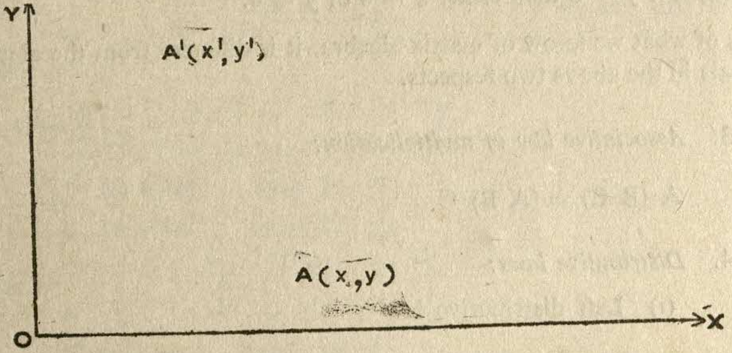
2·35 Some applications of matrices:

The applications of matrices, as has been pointed out in art 2·26, are numerous and wide. To be able to apply matrices to most of the more important concepts, however, a more thorough knowledge of all its aspects is necessary than can be provided in the scope of the discussion we shall be confining ourselves to. Given below are only some of the many applications of matrices.

Transformation:

Let (x, y) be the co-ordinates of a point A, (x′, y′) be the coordinates of another point A′ and let these coordinates be connected by the linear equations

$$x' = a_1 x + b_1 y$$
$$\text{and } y' = a_2 x + b_2 y.$$



where a_1, b_1, a_2, b_2 are real constants. These two equations provide a method of obtaining the co-ordinates of A′ from those of A, and may be written as

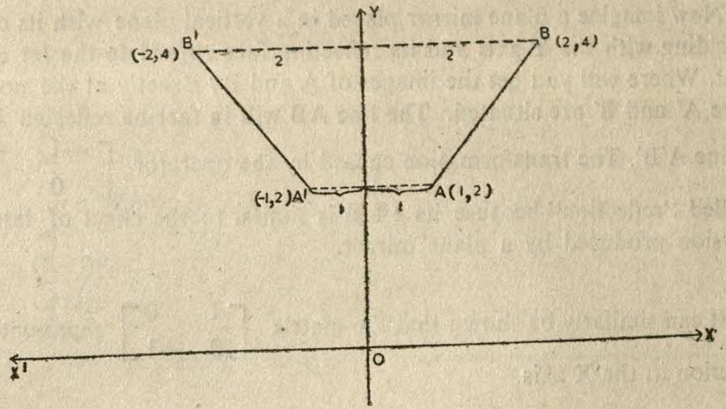
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

A′ is called the image of A.

The matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ represents the matrix which transforms the point A to the point A′. We shall call it the "*operator*". The nature of the transformation depends on the relative values of a_1, b_1, a_2, b_2. We consider below some of the transformations.

1. Reflection

Let the operator be $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$



Consider two points A(1, 2) and B (2, 4). Let images of A and B be A′ (x_1, y_1) and B′ (x_2, y_2) respectively.

Then, $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \therefore x_1 = -1, y_1 = 2$

∴ A′ is the point (—1, 2)

Similarly,

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \therefore x_2 = -2, y_2 = 4$$

∴ B′ is the point (—2, 4).

Note that the y co-ordinate of A′ is the same as that of A. Therefore A′ is situated at the same distance from X axis as A. Furthermore, the x coordinate of A′ is equal in magnitude but opposite in sign to that of A. Therefore A′ is situated at the same distance from Y axis as is A, but on oppsite side.

Similarly, the points B′ and B are situated at the same distance from X axis and B′ is situated at the same distance from Y axis as is B, but on opposite side.

Now imagine a plane mirror placed in a vertical plane with its edge coinciding with the Y-axis and its reflecting face turned to the 1st quadrant. Where will you get the images of A and B? Exactly at the points where A′ and B′ are situated. The line AB will in fact be reflected into the line A′B′. The transformation caused by the operator $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is called "reflection" because its effect is similar to the effect of lateral inversion produced by a plane mirror.

It can similarly be shown that the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ represents a reflection in the X axis.

2. Magnification:

Let the operator be $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. Consider the same points A (1, 2) and B (2, 4) as in the last case. Let A′ (x_1, y_1) and B′ (x_2, y_2) be the images of A and B respectively.

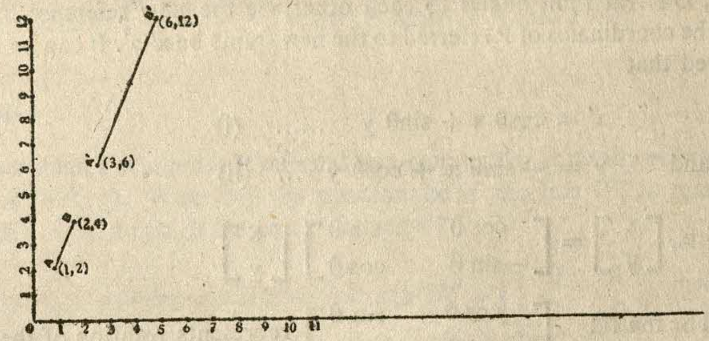
Then, $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$\therefore$ $x_1 = 3, y_1 = 6$. $\therefore$ A′ is the point (3, 6).

Similarly, $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$

$\therefore$ $x_2 = 6, y_2 = 12$. $\therefore$ B′ is the point (6, 12). Thus A (1, 2) is transformed to the point A′ (3, 6) and B (2, 4) is transformed to the point B′ (6, 12).

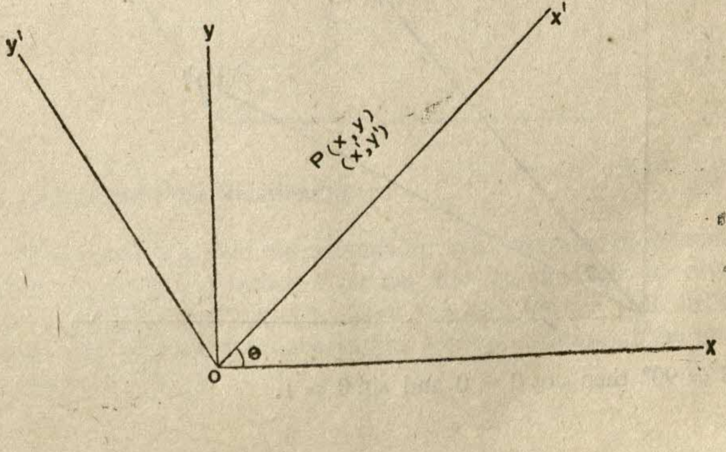
The line AB is transformed to the line A′B′. Evidently, the transformation causes a stretching or magnification in the original line AB.



If now the two operators of the last case and this case are combined in a single matrix, the operator $\begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}$ will cause a magnification of the line AB and also a reflection of it in the Y axis.

3. **Rotation :**

Let a point P have coordinates (x, y) referred to a rectangular cartesian reference frame OX, OY. Rotate the axes through an angle θ in *anticlockwise* sense and let



OX′, OY′ (at right angles to each other) be the new reference frame. Let the coordinates of P referred to the new frame be x′ y′. It can be easily proved that

$$x' = \cos\theta\, x + \sin\theta\, y \quad \ldots\ldots \quad \text{(i)}$$

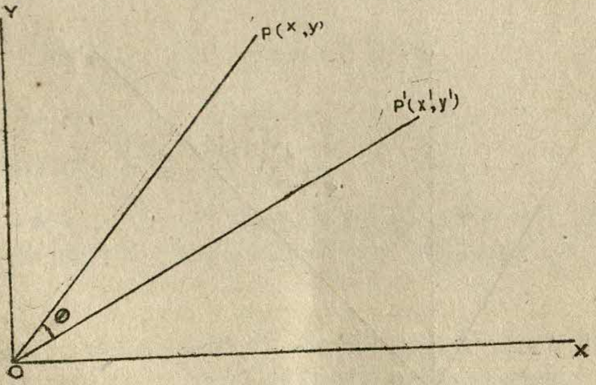
and $$y' = -\sin\theta\, x + \cos\theta\, y \quad \ldots \quad \text{(ii)}$$

i.e., $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ represents rotation of the *axes* through a positive angle θ. If the axes of reference remain fixed, and the line OP [where P is (x, y)] is rotated through an angle θ in the *clockwise* sense to the position OP′ so that the point P takes up the position P′ (x′, y′) then too, x′, y′ are given by the equations (i) and (ii). Thus the operator

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

causes a "rotation", essentially about the origin.



If θ = 90° then cos θ = 0 and sin θ = 1.

∴ The operator $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ causes a rotation through 90°.

Example :

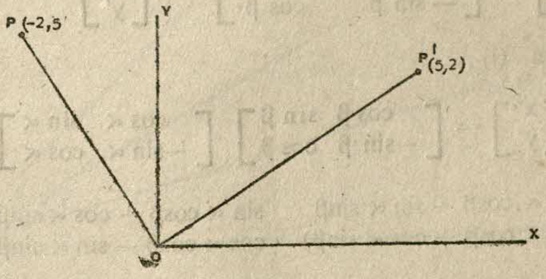
The coordinates of a point P referred to a rectangular cartesian reference frame is (—2, 5). What will its position be if the line OP is rotated through a right angle in the clockwise sense?

The operator causing a rotation through 90° is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Therefore, if $P'(x', y')$ be the new position of the point P, then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

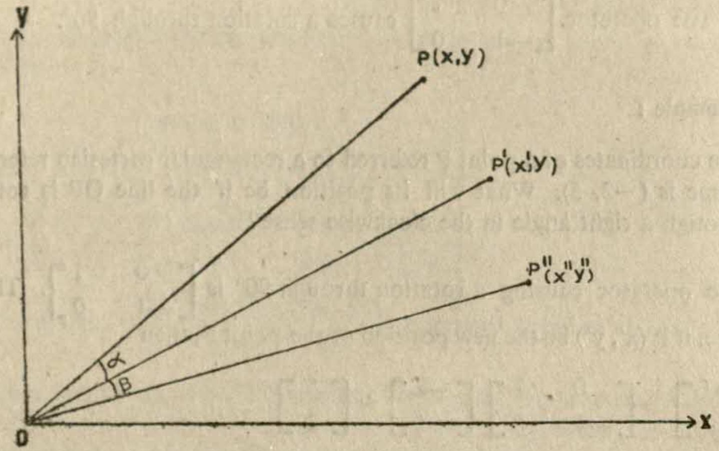
$$\therefore x' = 5, \quad y' = 2.$$

∴ the point P (—2, 5) takes up the position P′ (5, 2) when OP is rotated through 90° in the clockwise sense.



4. Trigonometrical Relationships:

Let P be a point (x, y) in the rectangular cartesian frame of reference. Let OP be turned in a clockwise direction through an angle ∝ so that P takes up the position P′ (x′, y′). Turn the line further still in the same direction through an angle β so that the final position of the point P becomes P″ (x″, y″).



We know that $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\propto & \sin\propto \\ -\sin\propto & \cos\propto \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ .... (i)

and, $\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$

∴ from (i),

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\propto & \sin\propto \\ -\sin\propto & \cos\propto \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \cos\propto\cos\beta - \sin\propto\sin\beta & \sin\propto\cos\beta + \cos\propto\sin\beta \\ -(\sin\propto\cos\beta + \cos\propto\sin\beta) & \cos\propto\cos\beta - \sin\propto\sin\beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{(ii)}$$

Now, since the angles ∝ and β are measured in the same direction, the line OP rotated through an angle (∝ + β) takes up the same position OP′. So,

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} \cos(\propto+\beta) & \sin(\propto+\beta) \\ -\sin(\propto+\beta) & \cos(\propto+\beta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ ....(iii)}$$

From (ii) and (iii)

$$\begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

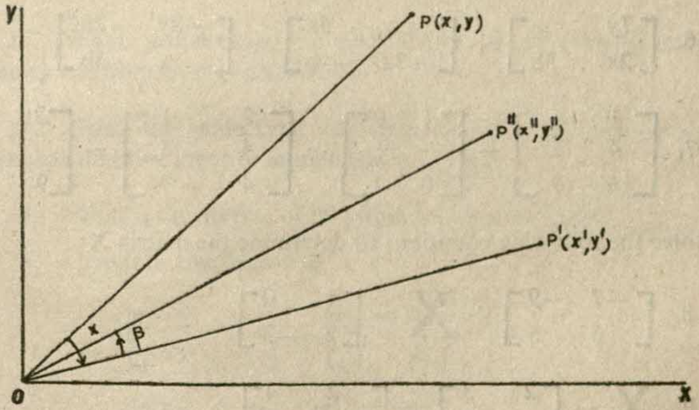
$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ -(\sin\alpha\cos\beta + \cos\alpha\sin\beta) & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix}$$

Therefore, by the definition of matrix equality,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \quad \text{and}$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

If the angle β is measured in the opposite direction, then by arguments exactly similar as above, it can be shown that

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \quad \text{and}$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$



It is left for the students to work out the proof in detail.

EXERCISE

Find the values of x, y, a, b from the following equations.

1. $\begin{bmatrix} 5x+2y & 2a-5b \\ 3x-4y & 5a+3b \end{bmatrix} = \begin{bmatrix} 46 & 23 \\ 12 & 11 \end{bmatrix}$

2. $\begin{bmatrix} a+2 & 18y-6 \\ 45x+15 & b-6 \end{bmatrix} = \begin{bmatrix} -10 & 4-21x \\ 24y+26 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 4b-y & 9x-8a \\ 12a+12x & 2b+3y \end{bmatrix} = \begin{bmatrix} 45 & 1 \\ 7 & 47 \end{bmatrix}$

Compute the following :

4. $\begin{bmatrix} -1 & 0 \\ 8 & 13 \end{bmatrix} + \begin{bmatrix} 11 & 9 \\ 5 & 4 \end{bmatrix}$

5. $\begin{bmatrix} \frac{1}{2} & \frac{5}{3} \\ \frac{7}{6} & \frac{1}{5} \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 10 & 1 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} & \frac{1}{3} \\ \frac{5}{6} & \frac{9}{5} \end{bmatrix}$

6. $\begin{bmatrix} 7y & a \\ 2x & 3b \end{bmatrix} + \begin{bmatrix} y & 5a \\ -3x & -b \end{bmatrix} + \begin{bmatrix} -2y & 2a \\ x & 3b \end{bmatrix}$

7. $\begin{bmatrix} \frac{1}{4} & \frac{5}{3} \\ 6 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{7}{4} & \frac{1}{3} \\ 4 & -7 \end{bmatrix} - \begin{bmatrix} 2 & 8 \\ 9 & 4 \end{bmatrix}$

Solve the following equations to determine the matrix X

8. $\begin{bmatrix} -7 & -9 \\ 6 & 5 \end{bmatrix} + X = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$

9. $X - \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}$

10. $\begin{bmatrix} 21 & \frac{6}{9} \\ -3 & 52 \end{bmatrix} - X = \begin{bmatrix} 21 & \frac{6}{9} \\ -3 & 52 \end{bmatrix}$

11. $X + \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$

12. $X + [0] = \begin{bmatrix} 4 & -5 \\ 6 & 1 \end{bmatrix}$

13. $X - [0] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

14. Is the equation $\begin{bmatrix} a & -\frac{1}{a} \\ c & k \end{bmatrix} + z = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$ valid ?

Justify your answer.

15. Does the summation $[-3 \quad 4] + \begin{bmatrix} 2 & 5 \\ \frac{1}{2} & \frac{6}{7} \end{bmatrix}$ make sense?

Why?

16. What properties does matrix addition have in common with ordinary addition in real numbers?

17. Using the same type of argument as in real number system, prove that additive identity in matrices is unique.

18. What is the inverse of [0] under matrix addition?

19. Compute the following

(i) $\begin{bmatrix} 2 & 0 \\ -1 & -3 \end{bmatrix} \times \begin{bmatrix} 6 & 1 \\ 0 & 4 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 9 \\ 5 & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} -8 \\ 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & -1 \\ 1 & 9 \end{bmatrix} \times \left\{ \begin{bmatrix} -2 & -3 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 10 & 8 \\ -1 & -2 \end{bmatrix} \right\}$

(iv) $\begin{bmatrix} \frac{2}{3} & \frac{4}{5} \end{bmatrix} \times \begin{bmatrix} 3 & 6 \\ 10 & 20 \end{bmatrix}$

(v) $\begin{bmatrix} \frac{1}{2} & \frac{1}{5} \end{bmatrix} \times \begin{bmatrix} \frac{2}{9} \\ \frac{5}{4} \end{bmatrix}$

20. Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Show that $AB \neq 0$ but $BA = 0$

21. Let $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}$

Show that $AB \neq BA$,

22. Show that for the matrix

$$A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} .$$

$A . A = 0$ whatever be the real values of a and b. Illustrate by taking particular values of a and b.

23. Show that whatever be the values of x, y, z and p,

if $A = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$ and $B = \begin{bmatrix} z & t \\ -t & z \end{bmatrix}$

then $AB = BA$.

24. Find the images of the points (0, 0), (0, 1), (2, 3), (– 4, 2), (5, –6) caused by the operator $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

25. If the axes are rotated through an angle 45° in the positive direction, find the changed coordinates of the points (1, 1), (–2, 1), (3, –2), (–5, –4).

26. P is the point (3, 7) in a reference frame with origin O. What will be the coordinates of the position which P takes up if OP

is rotated through an angle (i) 30°, (ii) 60°, (iii) 45°, (iv) 90°, (v) 180° in the clockwise sense?

27. If $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ be the operator of transformation, show in a diagram the position of O′P′, the image of the line OP where O and P are the points (0, 0) and (1, −3) respectively.

28. Find the reflection in the y axis of the points

$\left(2, \frac{1}{2}\right), \left(-\frac{1}{3}, 5\frac{1}{2}\right)$, (0, −8), (7, 0), (21, 6)

29. Find the reflection in the X axis of the points

(0, 0), (−3, 0), (−2, −5), $\left(0, \frac{1}{3}\right)$, (a, b)

30. Find to what points the four points

P (1, 0), Q (3, 0), R (3, 1) and S (1, 1) are transformed by the operator $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ If P′, Q′, R′, S′ are the images of P, Q, R, S respectively, draw a diagram and compare the figures PQRS and P′Q′R′S′. What do you find?

ANSWERS

1. $x = 8$, $y = 3$, $a = 4$, $b = -3$

2. $x = \frac{1}{3}$, $y = \frac{1}{6}$, $a = -12$, $b = 6$

3. $b = 13$, $y = 7$, $x = \frac{1}{3}$, $a = \frac{1}{4}$

4. $\begin{bmatrix} 10 & 9 \\ 13 & 17 \end{bmatrix}$

5. $\begin{bmatrix} -2 & 0 \\ -8 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 6y & 8a \\ 0 & 5b \end{bmatrix}$

7. $\begin{bmatrix} -1 & -6 \\ 1 & -5 \end{bmatrix}$

8. $\begin{bmatrix} 8 & 9 \\ -4 & -8 \end{bmatrix}$

9. $\begin{bmatrix} 4 & 7 \\ 3 & 2 \end{bmatrix}$

10. $[0]$

11. $\begin{bmatrix} 3 & \\ -2 & 2 \end{bmatrix}$

12. $\begin{bmatrix} 4 & -5 \\ 6 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

14. No.

15. No.

18. $[0]$

19. (i) $\begin{bmatrix} 12 & 2 \\ -6 & -13 \end{bmatrix}$

(ii) $\begin{bmatrix} -8 \\ -40 \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & -1 \\ 17 & 14 \end{bmatrix}$

(iv) $[10 \quad 20]$

(v) $\left[\frac{13}{36}\right]$

24. $(0, 0), (1, 0), (3, 2), (2, -4), (-6, 5)$

25. $\left(\sqrt{2}, 0\right), \left(-\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{5}{\sqrt{2}}\right) \left(-\frac{9}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

26. (i) $\left(\frac{3\sqrt{3}+7}{2}, \frac{7\sqrt{3}-3}{2}\right)$, (ii) $\left(\frac{3+7\sqrt{3}}{2}, \frac{7-3\sqrt{3}}{2}\right)$,

(iii) $(5\sqrt{2}, 2\sqrt{2})$, (iv) $(7, -3)$, (v) $(-3, -7)$.

28. $\left(-2, \frac{1}{2}\right), \left(\frac{1}{3}, 5\frac{}{2}\right), (0, -8), (-7, 0), (-21, 6)$.

29. $(0, 0), (-3, 0), (-2, 5), \left(0, -\frac{1}{3}\right), (a, -b)$

30. P′, Q′, R′, S′, are (1, 0), (3, 0) (4, 1) (2, 1) respectively. The figure PQRS suffers a ‘shear’ to the right.

CHAPTER THREE

Inequalities

3·1 In the previous chapter, we have discussed the properties of the real numbers, integers, rational numbers etc. These properties were studied from a viewpoint of the combination of two or more elements of each system under known operations, and the existence of some peculiar elements like identity, inverse which bear some specific properties when combined with all or some of the elements of the system under consideration. Throughout the discussion made, the notion of some specific operation was always involved. These properties or laws, however, do not enable us to compare two or more elements of any system. In this chapter, we wish to study the comparative properties of two (or more) elements of systems of numbers without any operation playing a role in the discussions. In other words, we wish to lay down some basic rules and definitions and then discuss from a comparative viewpoint the relation between any two elements of a system.

These basic rules are called axioms or postulates. They are to be taken for granted as the foundation on which the whole structure of the comparative study of numbers is erected.

3·2 The order axioms:

We begin by defining the relation "is greater than" denoted as "$>$" by the following axioms which are called the order axioms. In these axioms, as in the discussion throughout this chapter, a, b, c, d etc. represent any real numbers unless otherwise specifically mentioned. As usual, R represents the set of all real numbers.

A_1. The trichotomy axiom:

For any two real numbers a and b, that is, for any two elements a and b of R, one and only one of the following relations hold

$$a > b; \quad a = b; \quad b > a.$$

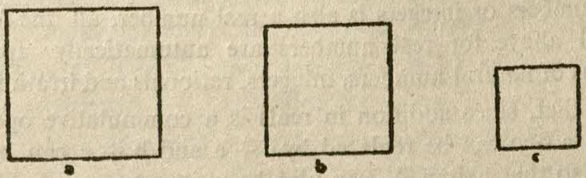
For example, consider the numbers 5 and 8. Out of the relations $5 > 8$; $5 = 8$; $8 > 5$ one is true, namely, $8 > 5$ and only this one is true, the other two are false.

Similarly, try with other pairs like $-4, 2$; $6, -\frac{1}{2}$; $0, 3$; $7, 7$ etc. and verify the axiom.

A_2. The transitive axiom :

For any three elements a, b, c of R, if $a > b$ and $b > c$ then $a > c$. For example, consider the three numbers -1, 1, 2. We have $2 > 1$ and $1 > -1$ and we know that $2 > -1$. Thus $2 > 1$ and $1 > -1$ leads to $2 > -1$.

This axiom is like thinking of three boxes a, b, c of different sizes.



If the box b can be put into the box a and the box c can be put into the box b, then the axiom guarantees that the box c can be put into the box a.

A_3. The axiom of addition :

For any three elements a, b, c of R, if $a > b$ then $a + c > b + c$. For example, we have $5 > 3$. Consider the relation

$$5 + \square > 3 + \square$$

What number, put in the box satisfies the inequality? Try with various numbers. You will find that the inequality holds whatever be the real number you choose to put into the box.

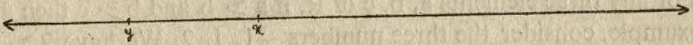
A_4. The axiom of multiplication:

For any three elements a, b, c of R, if $a > b$ and $c > 0$ then $ac > bc$. For example, we have $2 > -2$. Now put any positive real number in the box and verify that $2.\square > (-2)\square$.

Try similarly with (i) $5 > 1$ and (ii) $13 > 10$

We shall take the above four rules for granted, as constituting together, the definition of the relation "is greater than".

The relation "is greater than" in R conveys the same meaning as the idea "is to the right of" on the real line. For, if a point x is to the right of y on the real line,



then the real number corresponding to x is greater than the real number corresponding to y.

3·3. Since every element in the system of natural numbers or rational numbers or integers is also a real number, all the four axioms mentioned above for real numbers are automatically applicable to the systems of natural numbers, integers, rationals and irrationals.

Note that, since addition in reals is a commutative operation, in axiom A_3, $a + c$ can be replaced by $c + a$ and $b + c$ can be replaced by $c + b$ so that axiom A_3 can also be written as

"if $a > b$ then $c + a > c + b$."

Similarly, since multiplication in reals is a commutative operation, therefore in axiom A_4, ac and bc can be replaced by ca and cb respectively and hence it can also be written as "if $a > b$ and $c > 0$ then $ca > cb$".

3·4. The relation "is greater than" is called a binary relation, since it shows the relation between two elements. Care should be taken to observe and understand clearly the difference between a "binary operation" and a "binary relation." A binary operation, as discussed in the last chapter, is a definite mode of combination of two elements (at a time). When two elements of a system are combined by a binary operation, the result is a single element of the system. This process can be repeated and we can combine as many elements as we desire, the combination being made of two elements at a time; each time, we get a single element as a result of the combination and whatever be the number of elements combined, the ultimate result is a single element.

A binary relation such as "is greater than" does not convey the idea of combination of two elements to produce a third. It is a mode of comparing two elements of a system.

For example, the union in sets is a binary operation. Because, if A and B are two sets, $A \cup B$ is itself a set. The union "$\cup$" is therefore a mode of operation which combines two sets (at a time) to produce a single third set.

The subset notion $\subseteq$ is, however, a binary relation. For, it is used to compare two sets. If A and B are two sets then, $A \subseteq B$ shows a comparison or relation between the two sets.

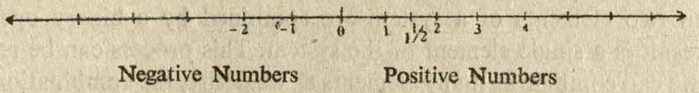
3·5. Sometimes, instead of making a comparison through the relation "is greater than", it is more convenient to think the other way round and make the comparison through the inverse relation "is less than". We make therefore, the following definition:

Definition 1. For any two elements a and b of R, a is said to be less than b (written as $a < b$) if and only if $b > a$.

Every axiom of Art. 3·2 can be written in terms of the relation "is less than". The axioms so formed constitute an independent set of axioms which may form the base of studying the comparative relationship between any two elements of a system of numbers.

3·6. Definition 2. If a real number 'a' is greater than zero then it is said to be "positive" and if it is less than zero then it is said to be "negative".

On the real line therefore, all the points to the left of zero represent negative numbers and all the points to the right of zero represent positive numbers.



3·7. Is zero positive or negative?

From axiom A_1, for any two real numbers a and b only one of

$$a > b, \quad a = b, \quad b > a$$

can be true. If a and b are both replaced by zero, then only one of

$$0 > 0; \quad 0 = 0; \quad 0 > 0 \quad \text{is true.}$$

But the numbers represented by the same symbol must be equal

$$\therefore \quad 0 = 0.$$

$\therefore$ $0 > 0$ is not true .... (i)

and $0 < 0$ is not true ....(ii)

from (i) 0 is not positive.

from(ii). 0 is not negative.

Thus, zero is neither positive nor negative.

3·8. We next proceed to prove some of the important properties of the inequalities in the form of the following theorems.

Theorem 1.

For any real number a, if $a > 0$ then $-a < 0$ and conversely.

Proof:

Since $a > 0$ $\therefore$ $a + (-a) > 0 + (-a)$ by A_3

$\therefore$ $0 > -a$

$\therefore$ $-a < 0$ def. 1.

Conversely, if $-a < 0$ then $0 > -a$ def. 1.

$\therefore$ $0 + a > (-a) + a$ by A_3

$\therefore$ $a > 0$

Theorem 2.

For any two real numbers a and b,

(i) $a > b$ if and only if $-a < -b$

(ii) $a < b$ if and only if $-a > -b$.

Proof (i) We shall first establish that

if $a > b$ then $-a < -b$.

Since $a > b$ $\therefore$ $a + (-a) > b + (-a)$ by A_3

$\therefore$ $0 > b + (-a)$

$\therefore$ $(-b) + 0 > (-b) + \{b + (-a)\}$ by A_3

$\therefore$ $-b > \{(-b) + b\} + (-a))$ associativity of addition

$\therefore$ $-b > 0 + (-a)$ inverse of addition

$\therefore$ $-b > -a$ identity of addition

$\therefore$ $-a < -b$ def. 1.

Next, we shall establish that if $-a < -b$ then $a > b$

Since $-a < -b$ $\therefore$ $-b > -a$ def. 1.

$\therefore$ $a + (-b) > a + (-a)$ by A_3

$\therefore$ $a + (-b) > 0$ identity of addition

$\therefore$ $\{a + (-b)\} + b > 0 + b$ by A_3

$\therefore$ $a + \{(-b) + b\} > b$ associativity

$\therefore$ $a + 0 > b$ inverse of addition

$\therefore$ $a > b$ identity of addition

Proof (ii) is left for the students to do, proceeding exactly along the same lines as Proof (i) .

Theorem 3.

If $a > b$ and $c < 0$ then $ac < bc$

Proof : Since $c < 0$ $\therefore$ $-c > 0$. (By theorem 1)

By axiom A_4, $a(-c) > b(-c)$

$\therefore \; -ac > -bc$

$\therefore \; ac < bc$ (By theorem 2.)

This theorem states that if both sides of an inequality is multiplied by a negative number then the sense of inequality is reversed.

e. g. : $5 > 3$

$\therefore$ By the above theorem, $5.(-7) < 3.(-7)$

$\therefore \; -35 < -21$

Note that axiom A_4 states that if both sides of an inequality is multiplied by a positive number then the sense of inequality remains unchanged.

Theorem 4.

If a and b are positive and $a > b$ then $\frac{1}{a} < \frac{1}{b}$.

Proof:

Since $a > b$ $\therefore$ $a \cdot \left(\frac{1}{a}\right) > b \cdot \left(\frac{1}{a}\right)$ By A_4 $\left(\because \frac{1}{a} > 0\right)$

$\therefore 1 > \frac{b}{a}$

$\therefore 1 \left(\frac{1}{b}\right) > \frac{b}{a} \cdot \left(\frac{1}{b}\right)$ By A_4 $\left(\because \frac{1}{b} > 0\right)$

$\therefore \quad \frac{1}{b} > \frac{1}{a}$

$\therefore \quad \frac{1}{a} < \frac{1}{b}$ def. 1

[That $a > 0$ implies $\frac{1}{a} > 0$ is left for the students to verify.]

Theorem 5.

For any real number a, $a^2 \geqslant 0$

Proof:

(i) If a is positive, $a > 0$

$\therefore \; a \cdot a > 0 \cdot a$ By A_4

$\therefore \; a^2 > 0$

(ii) If a is negative, $a < 0$ so that $-a > 0$.

Now, $0 > a$

$\therefore \; 0(-a) > a(-a)$ By A_4

$\therefore \; 0 > -a^2$

$\therefore \; a^2 > 0$ Theorem 1.

(iii) If a is zero, $a = 0$

$\therefore \; a(a) = 0(a)$

$\therefore \; a^2 = 0$

Thus, whatever be the real number a, $a \geqslant 0$. Note that $a^2 = 0$ if and only if $a = 0$.

Theorem 6.

For any two real numbers a and b, ab is positive if and only if a and b are either both positive or both negative.

Proof :

(i) 'If part' : Given ab is positive, to prove that a and b are either both positive or both negative.

Case I. Let $ab > 0$ and $a > 0$. We shall prove that $b > 0$.

Since $a > 0 \; \therefore \; \frac{1}{a} > 0$

Using $ab > 0$, we therefore have

$$ab\left(\frac{1}{a}\right) > 0\left(\frac{1}{a}\right)$$ By A_4.

$$\therefore \; a \cdot \frac{1}{a} \cdot b > 0 \cdot \frac{1}{a}$$ Commutativity of multiplication

$$\therefore \; b > 0$$

Thus a and b are both positive.

Case II. Now supose $ab > 0$ and $a < 0$. We wish to establish that $b < 0$.

Since $a < 0$. $\therefore -a > 0$

$$\therefore \frac{1}{-a} > 0$$

$$\therefore -\frac{1}{a} > 0$$

Now, $ab > 0 \therefore \left(-\frac{1}{a}\right) . a\,b > \left(-\frac{1}{a}\right) . 0$ By A_4

$$\therefore \left(-1\right) . \frac{1}{a} . a . b > 0$$

$$\therefore (-1)\,b > 0$$

$$\therefore -b > 0$$

$$\therefore b < 0$$

Thus a and b are both negative.

(ii) 'Only if' part:

Case I. Given a and b are both positive, to prove that ab is positive.

$a > 0 \therefore a\,(b) > 0\,(b)$ By A_4 ($\because b > 0$)

$$\therefore a\,b > 0$$

That is, ab is positive.

Case II. Given both a and b are ngative, to prove that ab is positive.

Since $a < 0 \therefore -a > 0$ Theorem 1 and def. 1.

Since $b < 0 \therefore (-a)\,b < 0\,(b)$ Theorem 3.

$$\therefore -a\,b < 0$$

$\therefore a\,b > 0$ def. 1.

That is, ab is positive.

EXERCISE

1. Sometimes the relation "is greater than" is defined as "$a>b$ if and only if $a-b$ is positive." Prove axioms A_2, A_3, and A_4 making use of this definition.
2. State if each of the following are true or false.
   (i) $0=-0$; (ii) $0(a)=0$; (iii) $0(-a)=0$ (iv) $a(-a)=a^2$.
3. Prove the following rule and illustrate it.
   If $a>b$ and $c>d$ then $a+c>b+d$.
   Which of the axioms did you use to prove it?
4. Prove that if a, b, c, d are positive real numbers, $a>b$ and $c>d$ then $ac>bd$.
   Give illustrations supporting the assertion.
5. In the previous example, a, b, c, d are positive real numbers. Does the law hold if this restriction is removed? Illustrate. Give an example of integers a, b, c, d to show that $a>b$, $c>d$ may imply $ac<bd$.
6. Using the rule proved in ex. 3 above, prove that if $a>b$ and $c>0$ then $a+c>b$.
7. Prove that
   (i) If $a>1$ then $a^2>a$.
   (ii) If a is positive and $a<1$ then $a^2<a$.
   Mention in each the axioms you use for the proof.
8. Extend the result (i) of the previous example and prove that if m and n are two positive integers such that $m>n$ and $a>1$ then $a^m>a^n$.
9. Use axiom A_3 to prove that if $a+c>b+c$ then $a>b$.
   [Hint: $a+c>b+c \therefore a+c+(-c)>b+c+(-c)$]
10. Use Ex. 3 to prove that
    (i) The sum of two positive real numbers is positive
    (ii) The sum of two negative real numbers is negative.
11. Use Ex. 4 to prove that if a and b are any two positive real numbers and $a>b$ then $a^2>b^2$.
    What axioms did you use to prove it?

12. If $a > b$, prove that $a > \frac{a+b}{2} > b$

[Hint: $a > b \therefore a + a > a + b \therefore 2a > a + b$

$\therefore \frac{1}{2}(2a) > \frac{1}{2}(a + b) \therefore a > \frac{a+b}{2}$

now prove the other part similarly.]

13. Use theorem 5 to prove that $1>0$.
14. With the help of the previous example prove that $2>1$ and hence that $2>0$. Which of the axioms did you use to prove the latter part?
15. Prove that $-3<-1$.
[Hint: $-3 < -2$ and $-2 < -1$]

ANSWERS

2. (i) True; (ii) True; (iii) True; (iv) False.
3. Axioms used are A_3 and A_2.
5. No.
Let $a = 3$, $b = -2$, $c = 1$, $d = -4$; then $ac = 1$, $bd = 8$ thus $a>b$, $c>d$ but $ac < bd$.
7. A_4
11. A_4, A_2
14. A_2.

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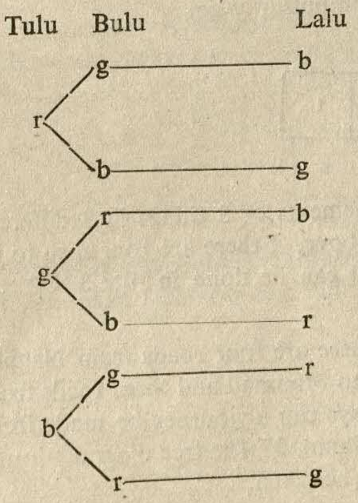
CHAPTER FOUR

Permutation and Combination, Binomial Theorem, Determinants, Surds and Indices

4·1. The Multiplication Principle:

Suppose you have three small wooden balls coloured red, blue, green respectively, and you wish to give them to three children Tulu, Bulu and Lalu, one each. In how many ways can you do this? The possible distributions are as shown below :



order of distribution

r g b

r b g

g r b

g b r

b g r

b r g

There are, obviously, six possible arrangements of 3 different balls among 3 boys. Note that the order of arrangement is important. Thus r b g and b g r constitute two different arrangements. The same result would be obtained if we consider the above problem as that of the arrangement of 3 different things in 3 positions.

3		

There are 3 ways of filling up the first position. Once the first position is occupied by one of the 3 things, there are now only two things left out.

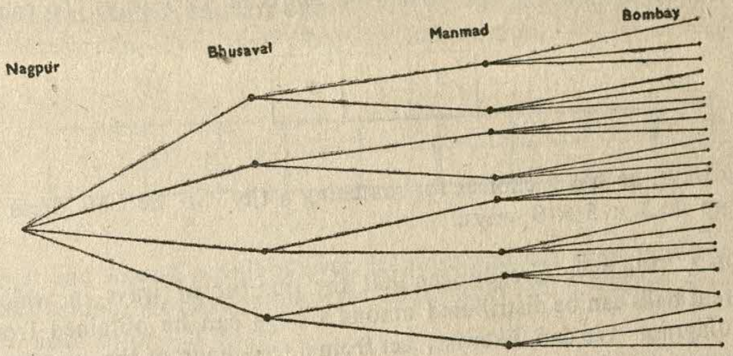
3	2	

So the second position can be filled in 2 ways. When the second position is also occupied, there is only one thing left out and only one position vacant, to which the last thing must compulsorily go.

3	2	1

There is therefore $3 \times 2 \times 1 = 6$ arrangements of 3 things in 3 different positions. With arguments similar as above, if there are four men to be seated in 4 chairs for a photograph, it can be done in $4 \times 3 \times 2 \times 1$ i.e., 24 ways.

Take another example. Suppose there are four roads from Nagpur to Bhusaval, two roads from Bhusaval to Manmad and three roads from Manmad to Bombay. In how many ways can a journey be made from Nagpur to Bombay via Bhusaval and Manmad? The tree diagram drawn below shows the possible routes that can be taken.



Count the number of possible arrivals at Bombay. You will find that it is exactly equal to $4 \times 3 \times 2$ i.e. 24.

In general, if a certain work can be done in p_1 ways, to each way of doing this work, a second work done in p_2 ways, to each way of doing the second work, a third work can be done in p_3 ways and so on, for n works, then the total number of ways in which all these works can be done together is $p_1 \times p_2 \times p_3 \cdots\cdots\cdots \times p_n$, where n is any +ve integral number. This is called the Multiplication Principle.

Example: A student has three trousers, five shirts, and 9 ties. In how many ways can he dress himself?

3		

The student can choose the trousers in 3 ways. To each choice of a trouser, he can make 5 choices of a shirt.

3	5	

Thus for 3 trousers, he can match his shirts in 3×5 ways. To each of

3	5	9

these ways, he has 9 choices for matching a tie. So he can dress him self up in 3 × 5 × 9 ways.

4·2. We have already seen that the number of ways in which 3 different balls can be distributed among 3 boys can be obtained from a tree diagram. We can, likewise, get from a tree-diagram the number of arrangements of the elements of any set of things in a given number of positions. The process, however, becomes very cumbersome when the numbers are very big. For example, would you like to draw a tree-diagram for the distribution of 25 different story books to 25 different boys of your class? The diagram, even if tried and drawn, will be uncannily long, and you would certainly wish to have some method of solving the problem other than by drawing the tree-diagram. We shall find, now, with the help of the multiplication principle, a general formula to deal with all such problems.

Let there be n different shaped mangoes, and suppose we wish to arrange them in n different positions. We shall, to start with, select any

1	2	3	4	—	—	—	—	n

one of the n fruits, and let it occupy the 1st position; this we can do in n ways. The first position filled, we come to the second. We are now left

n				–	– –	– –	– –

with (n—1) mangoes, and any one of them we can put to fill position 2. Thus there are (n—1) ways of filling the 2nd position.

n	(n-1)			--	---	---	

The first and second positions over, let us extend our considerations to the 3rd position. We have got now (n—2) mangoes left, so the 3rd position can be filled in (n—2) ways.

n	n-1	n-2				

Similarly, the 4th can be filled in (n—3) ways, the 5th in (n—4) ways, the 6th in (n—5) ways, and so on. (In how many ways can the 13th position be filled? Can you observe directly that the number of ways will be (n—12) without going through all the previous positions? How about the 20th position, 33rd position? r th position?)

n	(n-1)	(n-2)	(n-3)	---	n-(r-1)	---	2	1
1	2	3	4	---	r	---		n

As we proceed in this manner, the number of ways diminishes by one each time. Ultimately, when we come to the last position, that is the nth box, there will be only one mango left, so that the last position can be filled in only one way.

Making use of the multiplication principle, the total number of ways in which all these arrangements can be done is n(n—1) (n—2)......4.3.2.1. These arrangements, as we have already observed in article 4·1 have an order. That is, the place value, or the position value is important.

Such an arrangement is called a *Permutation*. A "permutation" thus is an ordered arrangement which depends on n, the number of things, and r, the number of positions to be filled by things chosen from among the n things. The permutation of n things taken r at a time is thus a functional relation of n and r, and therefore is denoted by P(n, r).

As discussed above, the permutation of n different things in n different positions, i. e., P(n, n) is n (n—1) (n—2) ......3.2.1.

4·3. Definition : The quantity n(n−1) (n—2) ....3.2.1. is for brevity, denoted as n! or (L^{n}) and is called "factorial n".

e. g.
$$5! = 5(5-1)(5-2)(5-3)(5-4) = 5\cdot4\cdot3\cdot2\cdot1 = 120.$$

$$7! = 7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1 = 5040.$$

$$m! = m(m-1)(m-2)(m-3)\ldots3\cdot2\cdot1.$$

note that $(n-1)! = (n-1)(n-2)(n-3)\quad\ldots3\cdot2\cdot1.$

and hence $n! = n\{(n-1)(n-2)(n-3)\ldots3\cdot2\cdot1\}.$
$$= n(n-1)!$$

Thus $(45)! = 45\,(44)!$

$(103)! = 103\,(102)!$ etc.

For reasons which will be obvious later, we define $0! = 1$

According to this definition, the permutation of n different things, taken all together (that is, to be arranged in n positions, so that none of the things are left out when all the positions are filled) which is denoted by $^{n}p_{n}$ or P(n,n) is n!

$$\therefore\ P(n, n) = n! = n(n-1)(n-2)\ldots3\cdot2\cdot1.$$

4·4. We shall find out next a formula for permutations when the number of positions is less than the number of given things:

Suppose there are seven different books and any three of them are to be placed in three different positions. The first position can be filled

7		

in 7 ways. For the second position, 6 books are left out, so that it can be filled in 6 ways.

7	6	

Lastly, 5 books having been left for the 3rd position, 5 ways are there to fill it. Hence by multiplication principle, the total number of arrange-

7	6	5

ments is 7.6.5. i.e. $\frac{7!}{4!}$ or $\frac{7!}{(7-3)!}$ ways.

In other words, $P(7,3) = 7.6.5. = \frac{7!}{(7-3)!}$

Similarly, the number of arrangements of 9 things taken 4 at a time (that is, in 4 different positions) is 9.8.7.6. i.e. $\frac{9!}{(9-4)!}$

Thus, P(7,3) is the product of three [Note the second component of (7,3) is 3] factors starting from 7 [i.e. the first component of (7, 3)], each factor being less by 1 than the previous factor. In the same way P(9,4) is the product of four factors starting from 9, each factor being less by 1 than the previous one. Find out P(6,2), P(5,3), P(8,5), thinking for them in the same way as above. In general, the number of arrangements of n different things taken r at a time is

$$\begin{aligned} P(n,r) &= n(n-1)\ (n-2) \ldots\ldots \text{ For r factors} \\ &= n(n-1)\ (n-2) \ldots\ldots \{n-(r-1)\} \\ &= n(n-1)\ (n-2) \ldots\ldots \{n-r+1\} \\ &= \frac{n\,!}{(n-r)\,!} \end{aligned}$$

Refer back to article 4·2. You will find that the number in the rth box is $n-(r-1)$ i.e., $n-r+1$; and if there were only r boxes to fill, the total number of arrangements, by multiplication principle, would be $n(n-1)\ (n-2)\ldots.(n-r+1)$, which is what we have now obtained as $P(n,r)$.

4·5. Examples:

1. How many words can be formed of the letters of the word "NAGPUR", by taking (a) three letters, (b) four letters, (c) six letters at a time?

There are 6 letters in the word NAGPUR.

(a) The number of arrangements taking 3 at a time is given by

$$P(6,3) = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1} = 120.$$

(b) $$P(6, 4) = \frac{6,\,!}{(6-4)\,!} = \frac{6\,!}{2\,!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot1} = 360.$$

(c) $$P(6, 6) = \frac{6\,!}{6-6)\,!} = \frac{6\,!}{0\,!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{1} = 720.$$

2. How many different words can be formed from the letters of the word "valedictory"? How many of them will begin with r? How many will begin with v and end with r?

$$\begin{aligned} \text{The total number of words} &= P(11, 11) = 11! \\ &= 11.\,10.\,9.\,8.\,7.\,6.\,5.\,4.\,3.\,2.\,1. \\ &= 39916800. \end{aligned}$$

For the number of words that begin with r, the first position is to be reserved for the letter r. Thus the arrangements have to be made for the other 10 letters in 10 positions.

∴ No. of arrangements which begin with r

$$= P(10, 10) = 10.9.8.7.6.5.4.3.2.1.$$
$$= 3628800.$$

For the words that begin with v, and end with r, the first and last (that is eleventh) positions are compulsorily occupied by v, and r respectively. Therefore, arrangements have to be made for the other 9 words that are left out among themselves.

∴ Number of words that begin with v, and end with r $= P(9,9) = 9! = 362880$.

3. In a basket are kept six fruits : a mango, a peach, a banana, an apple, an orange, and an apricot. Three children Tom, Mary and Dick are to be given one fruit each from the basket. (i) In how many ways can the distribution be arranged? (ii) In how many of these arrangements will the mango be always given away? (iii) In how many will the peach be always left in the basket?

(i) The number of arrangements of the fruits among the

$$\text{children} = P(6,3) = \frac{6\,!}{(6-3)\,!} = \frac{6\,!}{3\,!}$$

$$= \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1} = 120.$$

(ii) Since the mango is always to be given away, there are 5 fruits left out to be distributed to two children

Now if Tom gets the mango, the number of arrangements of the rest of the fruits is P(5,2). Similarly if Mary gets the mango, then again, there can be P(5,2) arrangements, and

Dick gets the mango, P(5, 2) arrangements correspond to it. Thus the total number of arrangements is

$$= 3 \times P(5, 2) = \frac{3 \times 5\,!}{(5-2)\,!}$$

$$= 3 \times \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1}$$

$$= 60.$$

(iii) The total number of arrangements in which the peach is always given away can be found as above to be equal to

$$3 \times P(5,2) = 60..$$

$\therefore$ The number of arrangements in which the peach is left in the basket = The total number of arrangements – the number of arrangements in which the peach is always given away.

$$= 120—60 = 60$$

Frequently, an indirect approach to a problem of permutation, as in this case, makes solving of it simpler.

The last part of the above problem, which can also be phrased as "In how many distributions will the peach never occur?" can be done in another way too. If the peach is never to be given away, there will be 5 fruits left out, to be given to 3 children. Hence the number of arrangements = P(5,3) = 5.4.3 = 60.

4·6. The Addition Principle :

If there are two mutually exclusive arrangements (i.e. two operations which cannot occur simultaneously) and if the first can be done in m ways, and the second in n ways, then the total number of arrangements will be m+n.

e. g. Suppose there are four triangular flags, coloured red, blue, green and white. How many signals can be made with them by using at least two of them simultaneously?

(i) If two flags are used, the number of signals = No. of arrangements of 4 things, 2 at a time. = P(4,2) = 12.

(ii) If three flags are used, number of signals =P(4,3) = 24.

(iii) If all four flags are used, number of signals = P(4,4) = 24.

$\therefore$ Total number of arrangements (i.e. signals)

= P(4,2) + P(4,3) + P(4,4)

= 12 + 24 + 24 = 60.

Here, the arrangements (i), (ii), (iii) are mutually exclusive. (that is, if two flags are hoisted to make a signal, at the same time 3 flags cannot be put up.)

So, the addition principle is made use of to combine the different kinds of permutations.

4·7. In all the above discussion of the permutations of various cases and kinds an important consideration is of the fact that the order of arrangements must be taken care of. To take a concrete example, if a red, a green and a white ball are to be permuted two at a time, then we have not only to select out all possible pairs like

$\begin{matrix} r & g \\ b & g \\ b & r \end{matrix}$; but to consider all possible arrangements in each selection again. Thus if the red and green balls are chosen, there will be wo arrangements $\left.\begin{matrix} rg \\ gr \end{matrix}\right\}$ corresponding to them. Similarly for the blue and green balls, there will be two independent arrangements among themselves, $\left.\begin{matrix} b & g \\ g & b \end{matrix}\right\}$ and so also for the selection of red and blue balls will there be two mutual arrangements within the selection itself given by $\left.\begin{matrix} b & r \\ r & b \end{matrix}\right\}$. The total number of arrangements thus being 6, we write P(3, 2) = 6.

Let us now turn our attention to problems in which we are interested merely in choosing out a number of things out of some given things, without caring for the order of arrangements. In the above discussion of the three balls, if we were asked merely to choose two balls out of the three (and once the choice is made, they need not be placed in any ordered position); we shall select either blue and green, or blue and red or red and green, the possible number of choice being

$$\left.\begin{matrix} r & g \\ r & b \\ b & g \end{matrix}\right\}$$

that is, 3. Note that, since the order is immaterial, once we choose out the red and green balls, this choice makes only one possibility and not two, like $\left.\begin{matrix} rg \\ gr \end{matrix}\right\}$.

Take another example. Suppose, in a inter-college debate competition three boys are to be sent. Four boys give their names for the competition. They are Deshmukh, Mitra, Rai and Verma. In how many ways can the selection be made? The selection can be done in 4 ways :

Deshmukh, Mitra, Rai
or Mitra, Rai, Verma
or Rai, Verma, Deshmukh
or Verma, Deshmukh, Mitra.

Note again, that once 3 boys are chosen out of the four, there is no question of any more possible arrangements among themselves. Thus Mitra, Rai, Verma will constitute only one selection whether we put down the names as Mitra, Rai, Verma or Rai, Verma, Mitra or Verma, Mitra, Rai etc. In other words, it is immaterial as to which name out of the three we write first and which afterwards.

In the same problem, if instead of merely selecting 3 boys out of 4, the question was of seating them in chairs numbered a, b, c then in each selection there would be some inter-arrangements. For, if Mitra, Rai, Verma are chosen, they have now to be seated in the chairs and this, as we have seen before, can be done in P(3,3) ways.

We therefore observe that an arrangement (permutation) is different from a selection. In fact an arrangement (i.e. permutation) is something more than a selection. For permuting 2 things out of 3 we first select the two and then in each selection make inter-arrangements. A selection in which no inter-arrangement is done, that is, in which order is not taken into account, is called a *Combination*. The total number of combinations of n things taken r at a time is written as C(n,r) which shows it is a functional value depending on n and r. ($r \leqslant n$).

In the case of the three balls, taken 2 at a time, the permutation is

$$P(3, 2,) = \frac{3!}{(3-2)!} = 6.$$

the combination is $C(3,2) = 3$ as we have actually found. So that
$P(3, 2) = C(3, 2) \times (2!)$

In the case of the four boys, taken 3 at a time, the number of permutations is $P(4,3) = 4.3.2. = 24$. The number of combinations is $C(4,3) = 4$ as we have actually found. So that

$$P(4, 3) = C(4, 3) \times (3!).$$

We next prove a theorem to establish a formula for combination.

4·8. Theorem: The number of combinations of a set of n dissimilar things taken r at a time is

$$C(n, r) = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r\cdot(r-1)\cdot(r-2)\ldots 3\cdot 2\cdot 1}$$

Proof : The number of ways in which r things can be selected from n dissimilar things is, by definition of combination, C(n, r). On making one of these selections, there are r things with us, and if an order value is desired, these r things can be arranged among themselves in r! ways. Thus each selection gives rise to r! arrangements. Since in all there will be C(n,r) selections, the total number of arrangements must be $C(n, r) \times r!$. But total number of arrangements of n things taken r at a time is denoted by P(n, r).

$\therefore$ $P(n, r) = C(n, r) \times (r)!$

$\therefore$ $C(n, r) = \dfrac{P(n, r)}{r!}$

But, $P(n, r) = \dfrac{n!}{(n-r)!} = n(n-1)(n-2)\ldots\ldots(n-r+1)$

$\therefore$ $C(n, r) = \dfrac{P(n, r)}{r!} = \dfrac{n!}{r!\,(n-r)!}$

and also, $C(n, r) = \dfrac{P(n, r)}{r!} = \dfrac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r(r-1)(r-2)\ldots\ldots 3\cdot 2\cdot 1}$

Hence the theorem is proved.

Corollary 1 : $C(n, r) = C(n, n-r)$

$$C(n, r) = \frac{n!}{r!\,(n-r)!}$$

$$\text{and } C(n, n-r) = \frac{n!}{(n-r)!\,(n-(n-r))!}$$

$$= \frac{n!}{(n-r)!\,r!}$$

$$\therefore \quad C(n, r) = C(n, n-r).$$

This formula helps to simplify calculations.

e. g. $C(17, 15) = C(17, 17-15)$

$$= C(17, 2) = \frac{17\cdot 16}{2\cdot 1} = 136$$

Corollary : $C(n, o) = C(n, n) = 1.$

4·9. Illustration: In how many ways can a committee of one teacher and five boys be formed out of 4 teachers and 10 boys ?

To form a committee, it is only the selection and not the order of selection that is to be considered. Therefore, the problem here is one of selection only (i.e. combination) and not of arrangement (that is permutation). The number of ways in which one teacher can be selected out of four $= C(4, 1)$ ...... (1)

The number of ways in which five boys can be selected out of ten $= C(10, 5)$ ...... (2)

To each selection in (1) there can be $C(10, 5)$ selections of the boys. Therefore, by multiplication principle,

The total number of ways in which the committee can be formed

$$= C(4, 1) \times C(10, 5)$$

$$= 4032.$$

4·10. Theorem: (Pascal's rule): If r and n are two natural numbers such that $1 \leqslant r \leqslant n$ then $C(n+1, r) = C(n, r\text{-}1) + C(n, r)$

Proof: We know from art. 4·8, that

$$C(n, r) = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r(r-1)(r-2)\ldots\ldots 3\cdot 2\cdot 1}$$

[Note that there are r factors in the numerator and the same in denominator.]

$$\therefore \quad C(n, r-1) + C(n, r)$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-r)}{(r-1)(r-2)\ldots\ldots 3\cdot 2\cdot 1} + \frac{n(n-1)(n-2)\ldots\ldots(n-r)(n-r+1)}{r(r-1)(r-2)\ldots\ldots\ldots\ldots 3\cdot 2\cdot 1}$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-r)[r+n-r+1]}{r(r-1)(r-2)\ldots\ldots\ldots\ldots 3\cdot 2\cdot 1}$$

$$= \frac{(n+1)\,n(n-1)(n-2)\ldots\ldots(n-r)}{r(r-1)(r-2)\ldots\ldots\ldots\ldots 3\cdot 2\cdot 1}$$

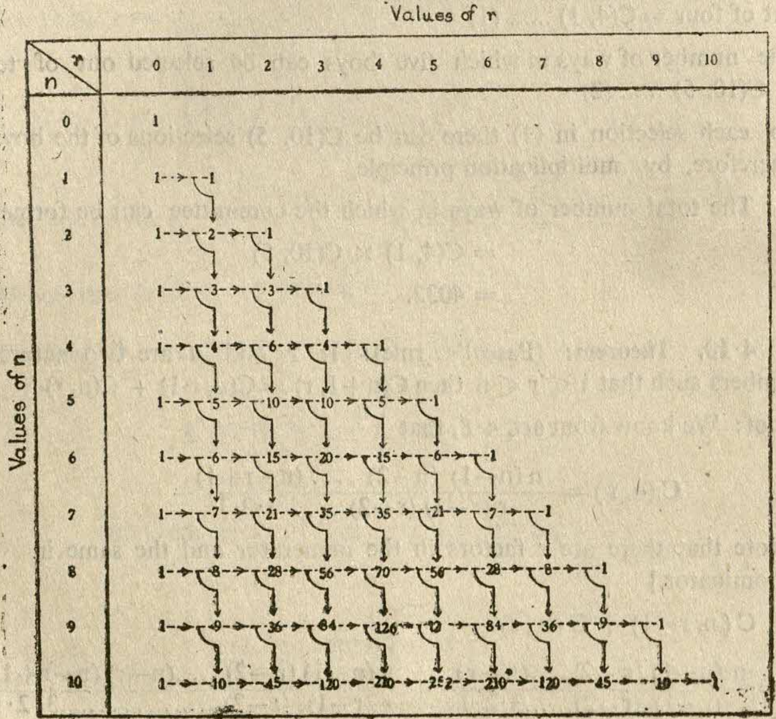
$$= C(n+1, r)$$

e. g. $C(9, 5) + C(9, 4) = C(10, 5)$.

4·11. Pascal's rule provides a simple way of building a table for the values of C(n,r). Shown below is the table for values of n upto 10

$$C\,(n+1,\, r) = C\,(n,\, r-1) + C\,(n,\, r)$$

$$1 \leqslant r \leqslant n$$



Values of r

Values of n

n \ r	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

This table, giving the values of C(n, r) which are arranged in a triangular form, is called Pascal's triangle. The first and last numbers in each row is 1 [because C (n, 0) = C (n,n) = 1]. For example, C(4, 0) given by the row corresponding to 4 (i.e., n = 4) and column corresponding to 0 i.e., (r = 0) is 1. So also C(4, 4) is given by the row corresponding to 4(n = 4) and column corresponding 4(r = 4) is 1. To write any other value of C(n, r) while forming the table, we have simply to add the values of combinations written in the previous row for r and $r-1$. For example, suppose, we have formed the table upto the row corresponding to $n=6$, and now wish to write down the next row corresponding to $n=7$. Consider any entry in this row, say C(7, 5). Since C(7, 5) is equal to C(6,4) + C(6,5), therefore it can be obtained by adding together the number in the previous row which is exactly above, and the number preceding it. So, C(7, 5) = 6 + 15=21. Similarly for all other entries in the table. Notice that the numbers in each row are symmetrically arranged about the middle number (or the pair of middle numbers). This is to be expected since

$$C(n, r) = C(n, n-r).$$

EXERCISES

1. A rat, being chased by a cat enters a room and finds 3 exits in the room. Each exit leads to two doors. In how many ways can the rat choose a way out of the room? Draw a tree-diagram to illustrate your answer.
2. How many pairs of the type (x, y) can be formed if x and y are to take values from among the elements of the set $\{\alpha, \beta, \gamma, \delta\}$ and $\{3, 4, 5\}$ respectively? Illustrate the set of all possible ordered pairs by a tree-diagram.
3. An urgent meeting of a committee of 30 members is to be called by the President. The president rings up four members and asks each of them to inform three members each of whom in their turn should contact two members to inform them about the meeting. How many members would be left out without the information? Illustrate your answer by a tree-diagram.
4. Find P(6, 2), P(20, 4), P(9, 5), P(8, 4), P(10, 3).
5. Find the values of : (i) $\frac{6!\,2!}{3!\,4!}$; (ii) $\frac{n!\,(2n)!}{(n-1)!^2}$;

   (iii) $\frac{P(10,3)}{7!}$.

6. Prove that $P(n, n-1) = P(n, n)$
7. Prove that $(2n)! = 2^n (n!). 1. 3. 5. 7. \ldots. (2n-1)$.
8. Show that $P(n+1, r) = (n+1) P(n, r-1)$.
9. How many different numbers can be formed with the digits 2, 3, 4, 9, 7 taken (i) two at a time, (ii) four at a time, (iii) all at a time?
10. In how many ways can a group photo of 9 men be arranged if any four of them must have the front row and five the back row?
11. How many words can be formed of the letters of the word "Hyperbola" consisting of 3 consonants and two vowels?
12. There are eight envelopes and eight greeting cards of different sizes. If four of the cards are too big for 3 of the envelopes, in how many ways can the cards be placed in the envelopes?
13. In how many ways can 3 letters be mailed in 5 mail boxes such that no two of them are mailed in the same box?
14. In a row of 12 chairs, 6 boys and 6 girls are to be seated alternately. In how many ways can this be done if the first chair is always to be occupied by one of the girls?
15. Find $C(3, 2)$, $C(6, 2)$, $C(10, 3)$, $C(32, 29)$, $C(102, 101)$, $C(54, 52)$, $C(535, 534)$.
16. Prove that

    (i) $r \cdot C(n, r) = n \cdot C(n-1, r-1)$ and

    (ii) $C(n, r+1) = \left(\frac{n-r}{r+1}\right) C(n, r)$; $0 \leqslant r < n$.

    Can Pascal's triangle be constructed by formula (ii) above ?
17. If $C(n+2, 4) = 6\, C(n, 2)$, find n.
18. Given 14 points, no three of which are collinear, find (i) the number of straight lines, (ii) the number of directed line segments that can be formed by joining pairs of these points.
19. In how many ways can four candidates be selected for a job for which twelve candidates apply?

20. A diagonal of a polygon is a straight line joining any two vertices of the polygon which are not adjacent. How many diagonals will a 16 sided polygon have? If a polygon has 44 diagonals, how many sides does it have?
21. There are 52 different cards in a pack of playing cards. If a hand is made up of 5 cards (disregarding the order) how many possible hands can be made?
22. In how many ways can a boy choose 2 gifts out of 8 different gift packets?
23. A committee of 4 boys and 2 girls is to be formed. In how many ways can the selection be made out of 15 boys and 7 girls?
24. A coach has 6 seats facing forward and 6 facing backward. In how many ways can 8 persons be seated in it if two of them refuse to face backward?
25. A student wants to choose three subjects out of Economics, Psychology, Chemistry, Physics, Botany, Zoology, Sanskrit and Mathematics. Find (i) In how many ways can he make his choice? (ii) In how many ways can he choose if Chemistry and Sanskrit classes are held at the same time? (iii) If he wants to take two science and one arts subject, how many combinations has he to make his choice from? (Consider Mathematics to be a science subject.)
26. There are two triple seated rooms, three double seated rooms and four single seated rooms in a hostel. In how many ways can 16 boys be accommodated in the hostel in each of the following cases?
    (i) 3 boys insist on having single rooms.
    (ii) two boys refuse to room together.
27. From 32 boys of a certain coy of N.C.C., a batch of 5 is selected every night to guard the campus. How many batches can thus be made? In how many of them will a particular boy (i) always be included. (ii) never be included?
28. How many triangles can be formed of twelve lines drawn in a plane, no two of which are parallel?

29. In an elocution competition, 7 boys and 5 girls wish to take part. In how many ways can their speeches be arranged if the competition must begin with a girl?

30. A person has 25 friends of whom 15 are men and the remaining ladies. In how many ways may he invite 18 friends, 10 of the invitees being men?

31. In a dinner party, 8 non-vegetarians and 6 vegetarians are to be seated in a row. In how many ways can this be done if one of the vegetarians refuses to have any non-vegetarian on either side of him?

32. In a chess tournament there are 8 participants. How many games must be arranged so that each participant plays with every other participant once and not more than once?

33. Three coins are tossed up into the air. Find the number of ways in which they may fall (one coin can fall with either head or tail turned to the looker).

    In how many of these will

    (i) one fixed coin always show head?

    (ii) one fixed coin never show head?

ANSWERS

1. 6
2. 12
3. 6
4. 30, 116280, 15120, 1680, 720.
5. (i) 10; (ii) $2^n . n^2 . (2n-1)(2n-3) \ldots\ldots 5 . 3 . 1$;
   (iii) $\frac{1}{7}$.
9. (i) 20, (ii) 120, (iii) 120.
10. 362880
11. 720
12. 2880
13. 60

14. 518400
15. 3, 15, 120, 4960, 102, 1431, 535
16. Yes
17. 7
18. (i) 91; (ii) 182.
19. 495
20. 104, 11
21. 2598960
22. 28
23. 28665
24. 4536000
25. (i) 56; (ii) 30; (iii) 30.
26. (i) 24(13)!; (ii) 222(14)!
27. 201376, 31465, 169911,
28. 220
29. 199584000
30. 135135
31. 9580032000
32. 28
33. 8; 4; 4.

4·12. Mathematical Induction:

Suppose, there are a number of metal rods and your friend challenges you to bend all of them. You take up one rod and bend it, and your friend agrees that you can bend them all. In this case,

(i) Your friend observes actually that you have bent the first rod.

(ii) He knows that if you can bend any one rod, you can bend the next rod.

Hence he is satisfied that your ability for bending the rods holds for all rods though he does not actually see you bending them all.

In mathematics an idea similar to this is sometimes used to prove certain assertions and the principle so formed is called the Mathematical Induction. If we wish to prove an assertion like $n(n+1)(n+2) = n^3 + 3n^2 + 2n$ by mathematical induction we shall do it in two steps.

(i) first, we observe actually that the assertion is true for one value of n, which is $n = 1$.

Here, if $n = 1$, $n(n+1)(n+2) = 1.(1+1)(1+2) = 6$.
and $n^3 + 3n^2 + 2n = 1^3 + 3\cdot1^2 + 2\cdot1 = 6$.

Hence $n(n+1)(n+2) = n^3 + 3n^2 + 2n$ is satisfied for one value of n, namely $n = 1$ ...... (1)

(ii) Next, we prove that if the assertion is true for one value of n, then it is true for the next value of n (that is, for the successor of n)

Let the assertion be true for the value r of n.

$$\therefore \quad r(r+1)(r+2) = r^3 + 3r^2 + 2r \ldots\ldots (2)$$

Adding $3(r+1)(r+2)$ on both sides,

$$r(r+1)(r+2) + 3(r+1)(r+2) = r^3 + 3r^2 + 2r + 3(r+1)(r+2)$$

$$\therefore \quad (r+1)(r+2)(r+3) = (r+1)^3 + 3(r+1)^2 + 2(r+1) \ldots (3)$$

that is, the assertion $n(n+1)(n+2) = n^3+3n^2+2n$ is true when $n = r+1$.

From (2) and (3) we observe that if the assertion is true for $n = r$, then it is true for $n = r + 1$......(4).

Now, from (1), the assertion is true for $n = 1$.

$\therefore$ from (4), the assertion must be true for $n = 1 + 1 = 2$. .... (5)

Again, from (5), the assersion is true for $n=2$.

$\therefore$ from (4), the assertion must be true for $n = 2 + 1 = 3$.

In this way, by taking the next number every time, it is established that the assertion is true for $n = 1, 2, 3, 4, 5, 6 \ldots$ i.e. for any +ve integral value of n. We therefore say that the assertion is generally true for positive integral values of n. With a motivation thus supported, we make the following rule :—

Statement : If P(n) represents an assertion and if

(i) The assertion is true when $n = 1$ that is, $P(1)$ is true, and

(ii) Whenever the assertion is true for a certain value of n say $n = m$, it is also true for the next value of n, i.e., $n = m+1$.

In other words, if $P(m)$ is true then $P(m+1)$ is true.

Then $P(n)$ is true for all positive integral values of n.

Note that the method of induction cannot be applied to prove any assertion for negative or fractional values of n.

4·13. Illustration:

Prove by Mathematical induction that

$$1 + 2 + 3 + 4 + \dots\dots\dots + n = \frac{n(n+1)}{2}$$

Let us denote the given assertion by $P(n)$

(i) First we have to prove that $P(1)$ is true. $P(1)$ is obtained by putting $n = 1$ in $P(n)$. $n = 1$ means there will be only one term on L. H. S. so that L. H. S. of the assertion is 1 when $n = 1$.

The R. H. S. of the assertion at $n = 1$ is $\frac{1(1+1)}{2} = 1$

Thus, $P(1)$ is $1 = \frac{1(1+1)}{2} = 1$ and this is true.

$\therefore$ $P(1)$ is true.

(ii) Next we assume that the assertion is true for some value m of n that is, $P(m)$ is true and we wish to show that it must be true for the next value of n that is, $P(m+1)$ is true.

$\because$ $P(m)$ is true.

$$\therefore \quad 1 + 2 + 3 + \dots\dots + m = \frac{m(m+1)}{2}$$

Adding $(m+1)$ on both sides,

$$\begin{aligned} 1 + 2 + 3 + \dots\dots m + (m+1) &= \frac{m(m+1)}{2} + (m+1) \\ &= (m+1)\left(\frac{m}{2} + 1\right) \\ &= \frac{(m+1)(m+2)}{2} \end{aligned}$$

which means $P(m + 1)$ is true.

Hence if $P(m)$ is true, $P(m + 1)$ must also be true.

$\therefore$ By the principle of induction the assertion is true for every positive integral value of n.

EXERCISE

By the method of mathematical induction, prove the following theorems for all +ve integral values of n :—

1. $2 + 4 + 6 + \dots\dots\dots + 2n = n(n + 1)$
2. $1 + 3 + 6 + \dots\dots\dots \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$
3. $2 + 2^2 + 2^3 + \dots\dots\dots 2^n = 2^{n+1} - 2$
4. $1.2 + 2.3 + 3{\cdot}4 + \dots\dots\dots + n(n + 1) = \frac{n(n+1)(n+2)}{3}$
5. $x^n - y^n$ is divisible by $x - y$
6. $1^2 + 2^2 + 3^2 + \dots\dots\dots + n^2 = \frac{n(n+1)(2n+1)}{6}$
7. $1^3 + 2^3 + 3^3 + \dots\dots\dots + n^3 = \frac{n^2(n+1)^2}{4}$
8. $a + ar + ar^2 \dots\dots\dots ar^n = \frac{a(1 - r^n)}{1 - r}$

Is the statement $n^2 + n = 2$ true for all +ve integral values of n?

[Hint: $P(1)$ is true but $P(m)$ does not imply $P(m + 1)$]

10. Is the statement $n^2 + (n + 1)^2 = 4(n + 1)$ true for all +ve integral values of n?

4·14. The Binomial Theorem :

For any +ve integer n, and any two numbers x and y

$$(x + y)^n = C(n, 0)\,x^n + C(n, 1)\,x^{n-1}y + C(n, 2)\,x^{n-2}y^2 + \dots\dots\dots + C(n, r)\,x^{n-r}y^r + \dots\dots\dots + C(n, n)\,y^n.$$

Proof : by mathematical induction

(i) If $n = 1$, L. H. S. of the assertion $= x + y$.

R. H. S. of the assertion $= C(1, 0)\, x + C(1, 1)\, x^0 y +$ terms in which the first component of combination is less than the second component.

But in a combination $C(n, r)$, if the first component is less than the second, it vanishes.

$\therefore$ R. H. S. of the assertion $= C(1, 0)\, x + C(1, 1)\, x^0 y$
$= x + y$

$\therefore$ L. H. S. of the assertion = R. H. S. of the assesrtion when $n = 1$.

i.e. the theorem is true when $n = 1$ —(1)

(ii) Let the theorem be true for some value, (say m) of n so that

$$(x + y)^m = C(m, 0)\, x^m + C(m, 1)\, x^{m-1} y + C(m, 2)\, x^{m-2} y^2 + \ldots\ldots + C(m, r)\, x^{m-r} y^r + \ldots\ldots + C(m, m)\, y^m.$$

multiply both sides of this equality by $(x + y)$.

$$\therefore\ (x+y)^{m+1} = (x+y)[C(m, 0)\, x^m + C(m,1)\, x^{m-1} y + C(m, 2)\, x^{m-2} y^2 + \ldots\ldots + C(m,r)\, x^{m-r} y^r + \ldots + C(m,m) y^m$$

$$= C(m, 0) x^{m+1} + C(m, 1)\, x^m y + C(m, 2) x^{m-1} y^2 + \ldots\ldots + C(m, r) x^{m-r+1} y^r + \ldots\ldots + C(m, m)\, y^m x$$

$$+ C(m, o) x^m y + C(m, 1) x^{m-1} y^2 + C(m, 2) x^{m-2} y^3 + \ldots + C(m, r) x^{m-r} y^{r+1} + \ldots\ldots + C(m, m)\, y^{m+1}$$

$$= C(m, o) x^{m+1} + [C(m, o) + C(m, 1)] x^m y + [C(m, 1) + C(m,2)] x^{m-1} y^2 + \ldots + [C(m, r-1) + C(m. r)]\, x^{m-r+1} y^r + \ldots + C(m, m)\, y^{m+1}$$

It has already been established that

$C(n, r-1) + C(n, r) = C(n+1, r)$.

$\therefore$ $C(m, o) + C(m,1) = C(m+1, 1)$

$C(m, 1) + C(m, 2) = C(m+1, 2)$ and so on.

In general, $C(m, r-1) + C(m, r) = C(m+1, r)$

$$\text{Also, } C(m, 0) = 1 = C(m+1, 0)$$
$$\text{and } C(m, m) = 1 = C(m+1, m+1)$$

$\therefore\ (x+y)^{m+1}=C(m+1, 0)x^{m+1}+C(m+1,1)x^{m}y+C(m+1, 2)x^{m-1}y^{2}$
$+\ldots+C(m+1, r)x^{m-r+1}y^{r}+C(m+1, m+1)\, y^{m+1}.$

This shows the theorem is true when $n = m + 1$.

So, if the theorem is true for $n = m$, it is also true for $n=m+1$..(2)
From (1) the theorem is true when $n = 1$.

$\therefore$ From (2), the theorem is true when $n = 1 + 1$
$= 2$ .....(3)

Again from (3) and (2) the theorem must be true when $n = 2 + 1 = 3$. Applying the argument successively, it is proved that the Binomial theorem holds for all +ve integral values of n.

4·15. The following properties of the Binomial theorem must be observed and remembered.

(1) The total number of terms is $n + 1$.

(2) The first term is $C(n, 0)x^{n}$, i. e. x^{n}; and the last term is $C(n, n)\, y^{n}$, i. e. y^{n}.

(3) In any term, the index of x is the difference between the components of the combination in that term; and the index of y is the second component of this combination.

(4) The index of x goes on decreasing by 1, and that of y goes on increasing by 1, from one term to the next term, the sum of the indices of x and y always remaining n.

(5) The coefficient of the first and last terms are equal, of the 2nd term from the beginning is equal to that of the 2nd term from the end; and in general, the coefficient of terms equidistant from the beginning and the end are same. This result we shall rigorously prove later.

Example : Expand $(a—b)^{6}$ by Binomial Theorem.
Use the Binomial Theorem, and in it, replace x by a, & y by (—b).

$$\begin{aligned}\therefore (a-b)^6 &= C(6,0)a^6 + C(6,1)a^{6-1}(-b)^1 + C(6,2)a^{6-2}(-b)^2 \\ &+ C(6,3)a^{6-3}(-b)^3 + C(6,4)a^{6-4}(-b)^4 + C(6,5)a^{6-5}(-b)^5 \\ &+ C(6,6)a^{6-6}(-b)^6. \\ &= a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6.\end{aligned}$$

Note that in the binomial expansion for the difference of two quantities, the terms are alternately positive and negative.

4·16. The General Term :

In the expansion for $(x+y)^n$, we observe that,

The first term i. e. $T_1 = C(n, 0)\, x^n = C(n, 0)\, x^{n-0}y^0$.

Second term i. e. $T_2 = C(n, 1)\, x^{n-1}y^1$

$T_3 = C(n, 2)\, x^{n-2}y^2$

$T_4 = C(n, 3)\, x^{n-3}y^3$

and so on.

So that $T_{r+1} = C(n, r)\, x^{n-r}y^r$.

the $(r+1)^{th}$ term in the binomial expansion is usually taken as the general term and is given by $T_{r+1} = C(n, r)\, x^{n-r}y^r$.

Example : Find the 17th term in the expansion of

$$\left(\frac{1}{x^2} - 5\, x^9\right)^{21}$$

We have, $T_{r+1} = C(21, r)\left(\frac{1}{x^2}\right)^{21-r}(-5\, x^9)^r$ ....(1)

Since we want the 17^{th} term, i.e. T_{17}, Put $r = 16$.

$$\therefore \text{from (1), } T_{17} = C(21, 16)\left(\frac{1}{x^2}\right)^{21-16}(-5\, x^9)^{16}$$

$$= C(21, 5)\,\frac{1}{x^{10}}(-5)^{16}.\, x^{144}\ [\because C(n, r) = C(n, n-r)]$$

$$= \frac{21\cdot 20\cdot 19\cdot 18\cdot 17}{5\cdot 4\cdot 3\cdot 2\cdot 1}\cdot 5^{16}\cdot x^{134}$$

$$= 20349\,(5)^{16}.\ x^{134}.$$

4·17. In any binomial expansion, the coefficients of terms equidistant from the beginning and the end are equal.

Let the binomial be $(x + y)^n$.

The $(r + 1)$th term from beginning $= C(n,r)\, x^{n-r} y^r$

$\therefore$ Coefficient of $(r + 1)$th term from beginning $= C(n, r)$

Consider now the $(r + 1)$th term from the end. Since there are $(n + 1)$ terms in all, the $(r + 1)$th term from end has $(n + 1) - (r + 1)$ i. e $n-r$ terms before it.

$\therefore$ $(r + 1)$th term from the end $= (n-r + 1)$th term from the beginning.

$$= T_{n-r+1}$$
$$= C(n, n-r)\ x^r\ y^{n-r}$$

$\therefore$ Coefficient of $(r + 1)$th term from the end

$= C(n, n-r)$

$= C(n, r)$

$=$ Coefficient of $(r + 1)$th term from beginning.

Example : Find the term independent of x in

$$\left(2x + \frac{1}{3x^2}\right)^9.$$

To find the term independent of x, we have actually to find that term in which the index of x is zero, that is, the term containing x°.

Let the $(r + 1)$th term contain x°.

here, $T_{r+1} = C(9, r)\ (2x)^{9-r} \left(\frac{1}{3x^2}\right)^r$

$$= C(9, r)\cdot 2^{9-r}\cdot \frac{1}{3^r}\ .\ x^{9-3r}$$

Since this contains x°. $\therefore$ the value of r must be such that the index of x is zero.

$\therefore\ 9-3r = 0\ \therefore\ r = 3.$

$\therefore$ the 4th term is independent of x, and it is given by

$$T_4 = C(9, 3)\cdot 2^{9-3}\cdot \frac{1}{3^3}\ x^{\circ}$$

$$= \frac{1792}{9}.$$

4·18. **Theorem :** In the expansion for $(1 + x)^n$

(i) The sum of coefficients of all the terms is 2^n.

(ii) The sum of the coefficients of even terms is equal to the sum of the coefficients of the odd terms.

By the Binomial theorem,

$$(1 + x)^n = C(n, 0) + C(n, 1)\,x + C(n, 2)\,x^2 + \dots\dots\dots\dots + C(n, n)\,x^n.$$

(i) Put $x = 1$

$$\therefore\ C(n, 0) + C(n, 1) + C(n, 2) + \dots\dots + C(n,n) = 2^n.$$

(ii) Put $x = -1$.

$$\therefore\ C(n, 0) - C(n, 1) + C(n,2) - C(n, 3) + \dots\dots + (-1)^n\, C(n, n) = 0$$

$\therefore$ C(n, 0) + C(n, 2) + C(n, 4) + ... + coefficients of odd terms = C(n, 1) + C(n, 3) + ... coefficients of even terms, and each

$= \frac{1}{2}$ [sum of coefficients of all terms]

$= \frac{1}{2}\cdot 2^n \quad = 2^{n-1}$.

EXERCISE

Expand the following binomials ;

1. $(a + 5)^4$
2. $\left(1 + \frac{3x}{2}\right)^7$
3. $\left(3y - \frac{2}{3}\right)^6$
4. $(x^{2/5} - 3y^{-2})^5$
5. $\left(x^2 + \frac{1}{x^3}\right)^4$

6. Find the ninth term of $\left(2 + \frac{x}{4}\right)^{17}$

7. Find the term containing x^4 in $(5-3x)^7$.

8. Find the term containing x^5 in $\left(6x - \frac{1}{3x}\right)^9$

9. Find the term independent of x in $\left(3x^2 + \frac{1}{6x^3}\right)^{15}$

10. Find the term independent of x in $\left(\frac{a}{x} + bx\right)^{20}$

11. Find the middle term in the expansion of $\left(3 + \frac{9}{x}\right)^{12}$

12. Find the middle terms in the expansion of $(6x^2 - y)^9$.

13. Find the 17th term in the expansion of
$\left(\frac{x^{3/2}}{a^{3/2}} - \frac{y^{7/2}}{b^{1/2}}\right)^{19}$

14. Find the term involving $\frac{x^2}{y^2}$ in the expansion of $\left(\frac{x}{y} - \frac{y^2}{2x^2}\right)^8$

15. Find the coefficient of x^{18} in $\left(x^2 + \frac{3a}{x}\right)^{15}$

16. If x^r occurs in the expansion of $\left(x + \frac{1}{x}\right)^m$ find its coefficient.

If C(n, o), C(n, 1) C(n, 2) ......are denoted by c_0, c_1, c_2 ....respectively, prove that

17. $c_1 + 2c_2 + 3c_3 + \ldots\ldots + nc_n = n \cdot 2^{n-1}$.

18. $2c_\circ + 2^2 \frac{c_1}{2} + 2^3 \frac{e_2}{3} + 2^4 \frac{c_3}{4} + \ldots 2^n \frac{c_n}{n+1} = \frac{3^{n+1}-1}{n+1}$.

Prove these by induction as well.

19. Show that
$[C(n, o)]^2 + [C(n, 1)]^2 + [C(n, 2)]^2 + \ldots\ldots + [C(n, n)]^2$
$= C(2n, n) = \frac{(2n)}{n\,!n\,!}$.

ANSWERS

1. $a^4 + 20\,a^3 + 150\,a^2 + 500\,a + 625$

2. $1 + \frac{21}{2}x + \frac{189}{4}x^2 + \frac{945}{8}x^3 + \frac{2835}{16}x^4 + \frac{5103}{32}x^5$
$+ \frac{5103}{64}x^6 + \frac{2187}{128}x^7.$

3. $729\,y^6 - 972\,y^5 + 540\,y^4 - 160\,y^3 + \frac{80}{3}y^2 - \frac{2}{27}y + \frac{32}{729}$

4. $x^2 - 15\frac{x^{\frac{8}{5}}}{y^2} + 90\frac{x^{\frac{6}{5}}}{y^4} - 270\frac{x^{\frac{4}{5}}}{y^6} + 405\frac{x^{\frac{2}{5}}}{y^8} - \frac{243}{y^{10}}$

5. $x^8 + 4x^3 + \frac{1}{x^2} + \frac{4}{x^7} + \frac{1}{x^{12}}$

6. $\frac{12155}{2^6}x^8$ 7. $2835\,x^4.$

8. $46656\,x^5$ 9. $\frac{1216215}{2^6}$

10. $184756\,a^{10}\,b^{10}$

11. $12012 \cdot \frac{3^{18}}{x^6}$

12. The middle terms are T_5 and T_6 and each $= 630 \cdot 6^5\,x^{10}\,y^4$

13. $969\left(\frac{x}{a}\right)^{\frac{9}{2}} \cdot \frac{y^{56}}{b^8}.$

14. $7 \cdot \frac{x^2}{y^2}$

15. $110565\,a^4.$

16. $\frac{m\,!}{\left(\frac{m-r}{2}\right)!\left(\frac{m+r}{2}\right)!}$

4·19. Determinants :

With each square matrix like $\begin{bmatrix} 6 & 3 \\ 5 & 4 \end{bmatrix}$ is associated an arrangement of the elements of the matrix written as

$$\begin{vmatrix} 6 & 3 \\ 5 & 4 \end{vmatrix}$$

which represents the real number $6 \cdot 4 - 5 \cdot 3$ that is, 9. Such an arrangement is called a "determinant". A *determinant* is, therefore, the real number thus formed associated with the corresponding matrix. In general, with the 2 × 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is associated the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$; and the value of this determinant is ad-cb. As in a matrix, a determinant consisting of two horizontal and two vertical lines is called a 2 × 2 determinant or a determinant of second order. The horizontal lines are called the rows, the vertical lines are called the columns, and each number in a row or a column is called an "element".

Example 1 : Find the real number corresponding to the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ under the determinant rule.

The determinant corresponding to this matrix is $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ which is equal to $1 \cdot 4 - 3 \cdot 2$, i. e. -2.

Thus the given matrix corresponds to the real number -2.

Example 2 : Solve the equation $\begin{vmatrix} 9 & 7 \\ x & 1 \end{vmatrix} = -5.$

since $\begin{vmatrix} 9 & 7 \\ x & 1 \end{vmatrix}$ is $9 \cdot 1 - x \cdot 7$ i e., $9 - 7x$, and this is given to be equal to -5; therefore we have $9 - 7x = -5$

$$\therefore \quad 7x = 14$$

$$\therefore \quad x = 2.$$

EXERCISE

Evaluate the following determinants:

1. $\begin{vmatrix} 4 & 6 \\ 3 & 7 \end{vmatrix}$ 2. $\begin{vmatrix} -2 & 5 \\ -1 & 4 \end{vmatrix}$

3, $\begin{vmatrix} 6 & -7 \\ 1 & -9 \end{vmatrix}$ 4. $\begin{vmatrix} -5 & 6 \\ 2 & -7 \end{vmatrix}$

5. $\begin{vmatrix} a & c \\ m & p \end{vmatrix}$ 6. $\begin{vmatrix} p & r \\ q & s \end{vmatrix}$

7. $\begin{vmatrix} y-4 & 2 \\ 3 & y \end{vmatrix}$

8. Solve the equation $\begin{vmatrix} x & 2 \\ 3 & 9 \end{vmatrix} = 0$ for x.

9. Solve for t : $\begin{vmatrix} -1 & t \\ 6 & -5 \end{vmatrix} = 3$

10. Solve for y : $\begin{vmatrix} -3 & 4 \\ y & 9 \end{vmatrix} = 2$

11. Prove that $\begin{vmatrix} a+b+2 & -b \\ a & o \end{vmatrix} = ab$

12. Prove that $\begin{vmatrix} 0 & x+4 \\ 0 & 3x^2 \end{vmatrix} = 0$

13. Prove that $\begin{vmatrix} x^2 & 12 \\ 3 & 1 \end{vmatrix} = (x-6)(x+6$

14. Solve for m : $\begin{vmatrix} m & -2 \\ 3\,m & 6 \end{vmatrix} = 12$

15. Solve for k : $\begin{vmatrix} 3 & -4k \\ k & -5 \end{vmatrix} = 1$

16. Find the value of r if $\begin{vmatrix} r & 3 \\ 3 & r \end{vmatrix} = 0$

17. Find the value of s if $\begin{vmatrix} s-2 & 10 \\ 2 & 3s \end{vmatrix} = 4$

18. Prove that $\begin{vmatrix} y-2 & y-2 \\ 3 & 5 \end{vmatrix} = 2(y-2)$.

ANSWERS

1. 10   2. -3 3. -47
4. 23   5. $ap-mc$,   6. $ps-qr$
7. y^2-4y-6
8. $\frac{2}{3}$
9. $\frac{1}{3}$
10. $-\frac{29}{4}$
14. 1
15. ± 2
16. ± 3
17. $-2, 4$.

4·20. Consider the two equations

$$py + q = 0 \;\ldots\ldots(1)$$
$$ry + s = 0 \;\ldots\ldots(2)$$

If there exists a value of y such that these two equations are satisfied by it simultaneously, then the two equations are said to be consistent. Now, if these two equations are consistent, then from (1) we have $y = \frac{-q}{p}$

Substituting this value for y in (2),

$$\left(\frac{-}{p}\right) + s = 0$$

$$\therefore -rq + sp = 0$$

$$\text{or} \quad ps - rq = 0$$

that is, $$\begin{vmatrix} p & q \\ r & s \end{vmatrix} = 0$$

This provides an easy formula for verifying whether or not two given linear equations are consistent.

Example 1 : Find the value of 'a' so that the equations $5x - 9 = 0$ and $ax + 3 = 0$ may be consistent.

If the two equations

$5x - 9 = 0$ and

$ax + 3 = 0$ are to be consistent,

then $\begin{vmatrix} 5 & -9 \\ a & 3 \end{vmatrix}$ must be equal to zero. So that $15 + 9a = 0$.

$$\therefore a = -\frac{5}{3}$$

Example 2 : Verify if the equations $3x + 12 = 0$ and $x + 6 = 0$ are consistent.

The equations $3x + 12 = 0$
and $x + 6 = 0$

are consistent provided

$$\begin{vmatrix} 3 & 12 \\ 1 & 6 \end{vmatrix} = 0$$

But this is not true. Hence the two given equations are not consistent.

4.21. Properties of determinants :

The following properties hold for any determinant of second order, and as you will see later, for determinants of higher orders also.

(1) If the rows of a determinant are changed into columns and columns are changed into rows, the value of the determinant remains unaltered.

Consider $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$; its value is ad−cb.

If, in this determinant, the rows are changed into columns and vice versa, then the determinant formed is $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ whose value is ad- bc, the same as that of the original determinant.

Compare with the fact that the matrix $\begin{pmatrix} a & d \\ c & d \end{pmatrix}$ is not the same as $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

(2) If each element of a row or a column of a determinant is multiplied by the same number then the determinant itself is multiplied by that number.

Consider the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$;

multiply each element of the first row by the constant k. The new determinant formed is

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = kad-kcb = k(ad-cb) = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

This rule can be equivalently stated as: "If the elements of a row (or a column) of a determinant have a common factor, then it can be taken out as a common factor for the whole determinant."

Example : Evaluate: $\begin{vmatrix} 20 & 16 \\ 27 & 18 \end{vmatrix}$.

By the rule 2 proved above,

$$\begin{vmatrix} 20 & 16 \\ 27 & 18 \end{vmatrix} = \begin{vmatrix} 4\cdot5 & 4\cdot4 \\ 9\cdot3 & 9\cdot2 \end{vmatrix}$$

$$= 4\cdot9\begin{vmatrix} 5 & 4 \\ 3 & 2 \end{vmatrix} = 36\cdot2\begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix} = 72\,[5-6] = -72.$$

(3) If two rows (or two columns) of a determinant are identical then the value of the determinant is zero.

Consider $\begin{vmatrix} a & a \\ b & b \end{vmatrix} = ab - ba = 0.$

(4) If two rows (or two columns) of a determinant are interchanged then the sign of the determinant changes but its absolute value remains the same.

Consider the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ whose value is $ad - cb$.

If now the two columns are interchanged, the new determinant formed is

$$\begin{vmatrix} b & a \\ d & c \end{vmatrix} = bc - ad = -(ad - cb)$$

$$= -\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

(5) If each element of a row (or column) of a determinant is the sum (or difference) of two terms then the determinant can be written as the sum (or difference) of two determinants.

Consider

$$\begin{vmatrix} a + k & b + m \\ c & d \end{vmatrix} = (a + k)d - (b + m)c$$

$$= ad + kd - bc - mc$$

$$= (ad - bc) + (kd - mc)$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} k & m \\ c & d \end{vmatrix}$$

Extending this law, we can similarly prove that

$$\begin{vmatrix} l_1a + m_1r & b \\ l_3c + m_3k & d \end{vmatrix} = \begin{vmatrix} l_1a & b \\ l_3c & d \end{vmatrix} + \begin{vmatrix} m_1r & b \\ m_3k & d \end{vmatrix}$$

4·22. Example. Evaluate $\begin{vmatrix} 12 & 16 \\ 13 & 17 \end{vmatrix}$

$$\begin{vmatrix} 12 & 16 \\ 13 & 17 \end{vmatrix} = \begin{vmatrix} 12 & 12 + 4 \\ 13 & 13 + 4 \end{vmatrix} = \begin{vmatrix} 12 & 12 \\ 13 & 13 \end{vmatrix} + \begin{vmatrix} 12 & 4 \\ 13 & 4 \end{vmatrix}$$

$$= 0 + [48 - 52] = -4.$$

A direct deduction of this property is a rule which is very often used to simplify calculations and is as follows :

"If the constituents of any row (or column) are added to or subracted from the corresponding constitutents of any other row (or column) then the determinant remains unaltered.

Note : While applying this rule, care must be taken to leave one row (or column) unaltered in one step.

4·23. Determinants of third order :

Just as a determinant of second order is a real number corresponding o a 2 × 2 matrix, to a 3 × 3 matrix of the type

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

corresponds a real number written as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

which is a determinant of third order representing the number

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

i. e., $a_1(b_2 c_3 - b_3 c_2) - b_1(a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2)$ ....(i)

The rule of expansion being "start with a_1, ignore the row and column passing through a_1, then form the determinant $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ with the remaining members. The product of a_1 with this determinant constitutes the first term of the expansion (i) above. Form similarly the second and third terms of the expansion starting with b_1 and c_1 respectively. Attach +ve sign to the first and third terms, and —ve sign to the second term." It can be seen that the real number obtained by expanding the determinant starting with the constituents a_1, a_2, a_3 of the first column (instead of the consitutents a_1, b_1, c_1, of the first row) with the same procedure as mentioned in the rule of expansion above, is same as (i). For, then the expression obtained would be

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

i. e., $a_1(b_2\,c_3 - c_2\,b_3) - a_2(b_1\,c_3 - c_1\,b_3) + a_3(b_1\,c_2 - b_2\,c_1)$.

and on opening the brackets, exactly the same expression as in (i) is obtained.

It is, therefore, immaterial whether a determinant of third order is expanded according to the elements of a row, or a column. The way of expansion for a particular problem is adopted in whichever way it is more convenient.

It is left for the students to verify that all the properties of a determinant that have been proved for a determinant of second order hold for a determinant of third order. In fact all these properties hold for higher order determinants too, but we shall be interested only in second and third order determinants.

Illustration : Evaluate $\begin{vmatrix} 1 & -1 & 1 \\ 4 & 2 & 10 \\ 2 & 2 & 6 \end{vmatrix}$

$$\begin{vmatrix} 1 & -1 & 1 \\ 4 & 2 & 10 \\ 2 & 2 & 6 \end{vmatrix} = 2\cdot 2 \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 5 \\ 1 & 1 & 3 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 0 & -1 & 1 \\ 3 & 1 & 5 \\ 2 & 1 & 3 \end{vmatrix} \quad \text{(adding the second column to the first)}$$

$$= 4\,[0(3-5) - 3(-3-1) + 2(-5-1)]$$

$$= 4\,[12-12] = 0.$$

EXERCISE

1. Find the value of x if

(a) $\begin{vmatrix} 2 & 4 & 8 \\ 1 & x & 5 \\ -3 & 1 & -2 \end{vmatrix} = 0$; (b) $\begin{vmatrix} 4 & 6 & -8 \\ 10 & 9 & -x \\ 2 & 1 & 3x \end{vmatrix} = 0.$

2. Verify without actual evaluation that the following determinants are equal.

$$\begin{vmatrix} 2 & 3 & 5 \\ -1 & 0 & k \\ x & 4 & \propto \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} -1 & x & \\ 0 & 4 & 3 \\ k & \propto & 5 \end{vmatrix}$$

(Hint : change rows into columns and columns into rows.)

3. Prove without actual evaluation that

$$\begin{vmatrix} 3 & -1 & 2 \\ 4 & 4 & 5 \\ 1 & -3 & -2 \end{vmatrix} = \begin{vmatrix} 4 & 1 & 3 \\ 4 & -3 & -1 \\ 5 & -2 & 2 \end{vmatrix}$$

4. Determine the roots of the equation

$$\begin{vmatrix} 3 & 1 & 3 \\ 2x & 2 & 2 \\ x^2 & 3 & 1 \end{vmatrix} = 0.$$

5. Without evaluating, prove that the value of each of the following determinants is zero.

(i) $\begin{vmatrix} 1 & -5 & 3 \\ 2 & -8 & 6 \\ 10 & 15 & 30 \end{vmatrix}$, ii) $\begin{vmatrix} 3 & 1 & -4 \\ 6 & 2 & -8 \\ 9 & 12 & 15 \end{vmatrix}$

6. Prove that if $a + b + c = 0$.

$$\begin{vmatrix} ax & by & cz \\ bz & cx & ay \\ cy & az & bx \end{vmatrix} = abc \begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}.$$

7. Show that $\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = 8\,abc$

8. Show that $\begin{vmatrix} 1 & 1 & 1 \\ x & 2 & 1 \\ x^3 & 8 & 1 \end{vmatrix} = (1-x)(x-2)(x+3)$

9. Show that $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$

10. Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b)$

11. Show that $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2\,abc\,(a+b+c)^3$

ANSWERS

1. (a) $\frac{27}{10}$ (b) $\frac{4}{5}$

4. 2, −3.

4·24. Laws of indices :

While dealing with the addition of algebraic expressions in which the same number occurs repeatedly, we write the result of addition as the product of the quantity and the number of times it repeats itself. For example,

$3 + 3 + 3 + 3$ is $4 \cdot 3$. i.e. 12, and in general,

$a + a + a + \ldots + a$.. (m times) is m.a.

We now set out to express in a convenient manner the products in which a quantity repeats itself a number of times.

Definition 1. If n is a positive integer, the product of n factors each equal to 'a' is represented by the symbol a^n called the "nth power of a" or "a raised to n".

That is, if n is a +ve integer, then a. a. a. ....n times $= a^n$

In the symbol a^n, a is called the base and n is called the exponent or the power.

$$\begin{aligned} \text{So, } 2^4 &= 2\cdot2\cdot2\cdot2 = 16 \\ 3^5 &= 3\cdot3\cdot3\cdot3\cdot3 = 243 \\ 10^7 &= 10\cdot10\cdot10\cdot10\cdot10\cdot10\cdot10 = 10000000. \end{aligned}$$

This definition gives rise to the following theorems.

Theorem 1 : If $a \varepsilon R$ then for any two positive integers m and n,

$$a^m \cdot a^n = a^{m+n}.$$

Proof : (by induction) :

Let P(n) denote $a^m \cdot a^n = a^{m+n}$

then P(1) is $a^m \cdot a^1 = a^{m+1}$

$$\begin{aligned} \text{Now,} \quad a^m \cdot a^1 &= (\,a\cdot a\cdot a\ldots\ldots m \text{ times}\,)\cdot a \\ &= a\cdot a\cdot a\ldots\ldots(m+1) \text{ times.} \\ &= a^{m+1}. \end{aligned}$$

$\therefore$ P(1) is true ......(i)

Assume that P(r) is true.

So, $a^m \cdot a^r = a^{m+r}$.

Multiply both sides by a.

$\therefore \quad a^m \cdot a^r \cdot a = a^{m+r} \cdot a$

$\therefore \quad a^m \cdot (a\cdot a\cdot a\ldots\ldots r \text{ times}).\ a = (a\cdot a\cdot a\ldots\ldots(m+r) \text{ times})\cdot a.$

$\therefore \quad a^m.(a\cdot a\cdot a\ldots(r+1) \text{ times}) = (a\cdot a\cdot a\ldots(m+r+1) \text{ times})$

$\therefore \quad a^m \cdot a^{r+1} = a^{m+r+1}$

$\therefore \quad a^m \cdot a^{r+1} = a^{m+(r+1)}.$

$\therefore$ P(r+1) is true.

Thus, P(r) implies P(r + 1) ....(ii)

From (i) and (ii), P(n), that is the assertion $a^m \cdot a^n = a^{m+n}$ is true for all positive integral values of n.

In the above proof, m is kept constant, and n is allowed to assume any positive integral value. Exactly similarly, we may keep n constant and allowing m to assume any positive integral value, prove the theorem. The theorem is therefore true for all +ve integral values of m and n. This result may be extended to any number of positive integral exponents, so that in general,

$$a^{m} \cdot a^{n} \cdot a^{p}\, a^{k} \ldots\ldots a^{l} = a^{m+n+p+k+\ldots+l}.$$

re m, n, p, k ......l is each a postive integer.

Examples :

(1) $5^3 \cdot 5^4 = (5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5 \cdot 5)$

$= 5^7,$

(ii) $$\frac{1}{2^2} \cdot \frac{1}{2^3} \cdot \frac{1}{2^4} = \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^4.$$

$$= \left\{\frac{1}{2} \cdot \frac{1}{2}\right\} \cdot \left\{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right\} \cdot \left\{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right\}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \text{ .... 9 times.}$$

$$= \left(\frac{1}{2}\right)^9 = \frac{1}{2^9}.$$

Theorem 2 : $(a^m)^n = a^{mn}$; $a \epsilon R$, m and n any positive integers.

Proof : $(a^m)^n = a^m \cdot a^m \cdot a^m$ .... n times.

$= a^{m+m+m \ldots}$n times.

$= a^{mn}.$

Example : $(2^3)^4 = 2^3 \cdot 2^3 \cdot 2^3 \cdot 2^3$

$= 2^{3+3+3+3}$

$= 2^{12}$

$= 2^{3 \cdot 4}.$

Theorem 3 : $(ab)^n = a^n b^n$, $a \in R$, $b \in R$ and n is a +ve integer.

Proof : $(ab)^n = (a \cdot b)\,(a \cdot b)\,(a \cdot b)$ ...... n times.

$= (a \cdot a \cdot a \ldots \text{n times})\,(b \cdot b \cdot b \ldots\ldots \text{n times})$

$= a^n \cdot b^n$.

Example : $(4 \cdot 5)^3 = (4 \cdot 5)\,(4 \cdot 5)\,(4 \cdot 5)$

$= (4 \cdot 4 \cdot 4)\,(5 \cdot 5 \cdot 5)$

$= 4^3 \cdot 5^3$.

Note that this is possible because associativity and commutativity in multiplication hold in the system of real numbers.

Theorem 4 : If $a \in R$ $(a \neq 0)$ and m and n are any two positive integers, then

$$\frac{a^m}{a^n} = \begin{cases} a^{m-n} & \text{if } m > n \\ \dfrac{1}{a^{n-m}} & \text{if } m < n \\ 1 & \text{if } m = n. \end{cases}$$

Proof : $\dfrac{a^m}{a^n} = \dfrac{a \cdot a \cdot a \ldots \ldots \ldots \text{m factors}}{a \cdot a \cdot a \ldots \ldots \ldots \text{n factors}}$

(i) if m>n then n factors each equal to 'a' will cancel out in numerator and denominator. Therefore the right side expression is reduced to a fraction in which the numerator contains the product of the number 'a' repeated m−n times and the denominator is 1.

$$\therefore \frac{a^m}{a^n} = \frac{a \cdot a \cdot a \ldots \ldots \ldots (m-n) \text{ factors}}{1}$$

$= a^{m-n}$.

(ii) if m<n then m factors each equal to 'a' cancel out in numerator and denominator. The right side expression is therefore reduced to a fraction in which the denominator is the product of 'a' repeated n−m times and the numerator is 1.

$$\therefore\ \frac{a^m}{a^n} = \frac{1}{a \cdot a \cdot a \ \ldots .(n - m) \text{ factors}}$$

$$= \frac{1}{a^{n-m}}$$

(iii) if $m = n$ then

$$\frac{a^m}{a^n} = \frac{a^m}{a^m} = 1$$

Examples :

(i) $$\frac{2^7}{2^4} = \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2}$$
$$= 2 \cdot 2 \cdot 2$$
$$= 2^3 \text{, i. e. } 2^{7-4}$$

(ii) $$\frac{2^3}{2^8} = \frac{2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}$$
$$= \frac{1}{2^5} \cdot \text{i. e. } \frac{1}{2^{8-3}}$$

(iii) $$\frac{2^5}{2^5} = \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}$$
$$= 1$$

4·25. We have discussed above the laws of exponents when the exponent or index is a positive integer. Before extending the rules to all integers, it is necessary that a meaning should be attached to a real number raised to an index other than a positive integer. For, all we understand at this stage about an expression like a^m is the product of 'a' repeated m times and this is meaningful only when m is a + ve integer. If m is a —ve integer, there is no meaning in saying "the product of 'a' repeated m times", for we cannot possibly multiply a number to itself a negative number of times.

[e. g. $2^7 = 2 \cdot 2. \ldots .7$ times is meaningful, but can you say $2^{-7} = 2 \cdot 2 \ldots .(-7)$ times? Obviously not, for the number of repetitions has got to be a +ve integer.] We need to attach, therefore, a specific meaning to a^m where m is either zero or a negative integer, and it should be a meaning correlating to that of definition 1, of art. 4·24. To this end we make the following definitions.

Definition 2 : For any $a \in R$ $(a \neq 0)$

$$a^\circ = 1$$

Definition 3 : For any $a \in R$ $(a \neq 0)$

$$a^{-n} = \frac{1}{a^n}$$

where n is any positive integer.

With these definitions, the rules of exponents can now be extended to the negative and zero exponents, as in the following theorem :

Theorem 5 : For any $a \in R$ $(a \neq 0)$ and any $m, n \in I$.

(i) $a^m \cdot a^n = a^{m+n}$

(ii) $(ab)^n = a^n b^n$

(iii) $(a^m)^n = a^{mn}$

(iv) $\dfrac{a^m}{a^n} = a^{m-n}$

Proof :

(i) (a) If m and n are +ve integers, then $a^m \cdot a^n = a^{m+n}$ is already proved.

(b) Let m be a +ve integer and n be a −ve integer say $n = -p$ where p is positive.

$$\text{Then } a^m \cdot a^n = a^m \cdot a^{-p} = \frac{a^m}{a^p} = a^{m-p}$$

$$= a^{m+(-p)} = a^{m+n}$$

(c) Let m and n be both −ve integers.

Let $m = -p$, $n = -q$, where p and q are +ve integers

$$\text{Then } a^m \cdot a^n = a^{-p}\, a^{-q} = \frac{1}{a^p} \cdot \frac{1}{a^q}$$

$$= \frac{1}{a^{p+q}} = \frac{1}{a^{-(m+n)}} = a^{m+n}$$

The proofs of the other parts are left to the students as exercises. They can be obtained very easily by making use of the two definitions given above and Theorems 1 through 4 of the last article.

Example 1 : Simplify $\frac{5x^7\,y^{-3}}{2x^{-2}\,y^5}$ by removing the negative and zero exponents.

$$\frac{5x^7\,y^{-3}}{2x^{-2}\,y^5} = \frac{5}{2} \cdot \frac{x^7}{x^{-2}} \cdot \frac{y^{-3}}{y^5}$$

$$= \frac{5}{2} \cdot x^{7-(-2)} \cdot y^{-3-(5)}$$

$$= \frac{5}{2} \cdot x^9 \cdot y^{-8}$$

$$= \frac{5}{2}\,\frac{x^9}{y^8} = \frac{5x^9}{2y^8}$$

Example 2 : Find the value of $\frac{(-3)^5\,(2)^{-4}}{3^7\,(-2)^{-9}}$

$$\frac{(-3)^5\,(2)^{-4}}{3^7(-2)^{-9}} = \frac{\{(-1)\cdot 3\}^5 \cdot 2^{-4}}{3^7 \cdot \{(-1)\cdot 2\}^{-9}}$$

$$= \frac{(-1)^5 \cdot 3^5 \cdot 2^{-4}}{3^7 \cdot (-1)^{-9} \cdot 2^{-9}}$$

$$= (-1)^{5+9} \cdot 3^{5-7} \cdot 2^{-4-(-9)}$$

$$= (-1)^{14} \cdot 3^{-2} \cdot 2^5$$

$$= 1 \cdot \frac{1}{3^2} \cdot 2^5$$

$$= \frac{32}{9} \quad [\because (-1)^{14} = 1]$$

4·26. We next proceed to discuss the laws of exponents when the indices are any rational numbers. We begin by making the following definition.

Definition 4 : If n is a positive integer and a and b are any two numbers such that $a^n = b$, then b is called the nth power of a, and a is called the nth root of b.

Equivalently, we may say that any number a is called the nth root of b provided the nth power of a is b.

For example, Since $3^2 = 9$ that is, since the square of 3 is 9 therefore the square root of 9 is 3 (written as $\sqrt{9} = 3$); also, since thesquare of -3 is 9 therefore the square root of 9 is -3 (written as $\sqrt{9} = -3$). Thus 3 and -3 are the two square roots of 9.

Similarly since $(-2)^3 = -8$ that is, since cube of -2 is -8, therefore one cube root of -8 is -2. (Is there any other cube root of -8?)

It is obvious that if $a^n = b$, then 'a' is one of the nth roots of 'b'. There may be other nth roots of b; for, there may exist other numbers whose nth power equals b. In fact, though we shall not undertake it within the scope of the present discussion, it can be proved that if $a^n = b$ then b has n distinct roots (some of which may be outside the system of real numbers). It is deemed convenient, therefore, to decide to attach a particular meaning to the notation we shall shortly adopt for a root. With this view, we define the principal nth root.

Definition 5 : The principal nth root of a positive number is the positive root of the number. The principal nth root of a $-$ve number is its $-$ve root if n is odd. If n is even, the principal nth root of a $-$ve number is undefined within the system of real numbers.

Throughout the discussion made hereafter the notation $\sqrt[n]{b}$ will stand for the principal nth root of b, and unless otherwise specifically stated, the nth root of b will mean the principal nth root of b. Thus, $\sqrt[4]{16} = 2$ (not -2, though $(-2)^4$ is also equal to 16). Also, the square root of any number b will be written as $\sqrt{b}$. The symbol $\sqrt{\ }$ is called a radical. If $\sqrt[n]{b} = a$ then n is called the index of a (for it is the power or exponent of a). n is also sometimes called the index of the radical; b is called the radicand of the radical. Before stating finally the rules of exponents in rational numbers, one more definition is in order.

Definition 6. A q^{th} root of b is written as $b^{1/q}$ and we define it such that $(b^{1/q})^p = b^{p/q}$.

By this definition and the notation already adopted, we now have

$$b^{p/q} = (b^p)^{1/q} = \sqrt[q]{b^p}$$

and also, $b^{p/q} = (b^{1/q})^p = \left(\sqrt[q]{b}\right)^p$.

So that $\sqrt[q]{b^p} = \left(\sqrt[q]{b}\right)^p$

4·27. With this notion of a fractional index, we are now equipped to state the laws of exponents for rational numbers in the form of the following theorem:

Theorem 6 : For any $a \epsilon R$ and any rational numbers m and n,

(i) $a^m . a^n = a^{m+n}$.

(ii) $(ab)^n = a^n b^n$

(iii) $(a^m)^n = a^{mn}$

(iv) $\dfrac{a^m}{a^n} = a^{m-n}$

Proof : (i) $a^m . a^n = a^{m+n}$.

Case I : If m and n are integers then the rule stands by the previous discussion.

Case II : Let m be a proper fraction (that is a fraction which is a ratio of two integers, and which cannot be reduced to an integer) and n an integer.

Let $m = \dfrac{p}{q}$, where p and q are integers and $q \neq 0$.

$$\begin{aligned}
a^m \cdot a^n &= a^{p/q} . a^n \\
\therefore (a^m \cdot a^n)^q &= (a^{p/q} . a^n)^q \\
&= [\,(a^{1/q})^p . a^n)\,]^q && \text{(from def. 6)} \\
&= (a^{1/q})^{pq} . a^{nq} \\
&= a^{\frac{pq}{q}} . a^{nq} \\
&= a^p . a^{nq} \\
&= a^{p+nq} && \text{(from the law of exponents for integers.)}
\end{aligned}$$

$$\begin{aligned}
\therefore \quad a^m . a^n &= \sqrt[q]{(a^{p+nq})} \\
&= (a^{p+nq})^{1/q} \\
&= a^{\frac{p+nq}{q}} \\
&= a^{p/q+n} \\
&= a^{m+n}.
\end{aligned}$$

Case III : Let m and n be proper fractions.

Let $m = \dfrac{p}{q}$ and $n = \dfrac{r}{s}$ where p, q, r, s are integers and $q \neq 0$, $s \neq 0$.

Let k be the L. C. M of q and s so that k is divisible by q as well as s, and assume that $k = bq$ and $k = cs$ where b and c are integers.

$$\begin{aligned}
a^m . a^n &= a^{p/q} . a^{r/s} \\
\therefore (a^m \cdot a^n)^k &= (a^{p/q} . a^{r/s})^k \\
&= [\,(a^{1/q})^p . (a^{1/s})^r\,]^k \\
&= (a^{1/q})^{pk} . (a^{1/s})^{rk} \\
&= a^{\frac{pk}{q}} \cdot a^{\frac{rk}{s}} \\
&= a^{pb} . a^{rc}
\end{aligned}$$

$$\begin{aligned}
\therefore \quad a^m . a^n &= (a^{pb} . a^{rc})^{1/k} \\
&= (a^{pb+rc})^{1/k}
\end{aligned}$$

$$= a^{\frac{pb+rc}{k}}$$

$$= a^{\left(\frac{pb}{k}+\frac{rc}{}\right)}$$

$$= a^{\left(\frac{p}{q}+\frac{r}{s}\right)} \quad \left(\because \frac{b}{k}=\frac{1}{q} \text{ and } \frac{c}{k}=\frac{1}{s}\right)$$

$$= a^{m+n}.$$

All the other laws of the theorem can similarly be proved and are left as exercises to the students.

4·28. Surds : We have already seen that the nth root of any real number b is called a radical of b of n^{th} order. It may so happen that the real number b cannot be expressed as the n^{th} power of any rational number, but is the n^{th} power of an irrational number. Then, the n^{th} root of b is an irrational number. In such a case, the radical is called a **surd.**

If the radical is of order two, the surd is called quadratic; if the radical is of order three, the surd is called cubic and so on.

e.g. $\sqrt{5}$ is a quadratic surd.

$\sqrt[3]{7}$ is a cubic surd.

$\sqrt[4]{9}$ is a quartic surd.

Pure and mixed Surds :

A surd consisting wholly of an irrational number is called a pure surd, and a surd consisting of a rational and an irrational number is called a mixed surd.

e.g. $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[4]{9}$ is each a pure surd. $2+\sqrt{3}$, $3\sqrt[5]{7}$, $8\sqrt[3]{10}$ are mixed surds. The laws of radicals follow directly from the previous definitions and theorems and are as following :

If m and n are positive integers, then

1. $(\sqrt[n]{a})^n = a$
2. $\sqrt[n]{ab} = (\sqrt[n]{a})(\sqrt[n]{b})$

3. $\left(\sqrt[n]{\frac{a}{b}}\right) = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

4. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Illustrations :

(i) $\sqrt[3]{36} \div \sqrt[3]{48} = \sqrt[3]{\frac{36}{48}}$

$$= \sqrt[3]{\frac{3}{4}}$$

(ii) $\sqrt[4]{\frac{169\ x^6z^2}{y^4}} = \frac{\sqrt[4]{169\ x^6z^2}}{\sqrt[4]{y^4}}$

$$= \frac{\sqrt[4]{169}\ \sqrt[4]{x^6}\ \sqrt[4]{z^2}}{y}$$

$$= \frac{\sqrt{\sqrt{169}} \times \sqrt{\sqrt{x^4x^2}} \times \sqrt{\sqrt{z^2}}}{y}$$

$$= \frac{\sqrt{13} \times \sqrt{x^2x} \times \sqrt{z}}{y}$$

$$= \frac{\sqrt{13} \times \sqrt{x^2\,x} \times \sqrt{z}}{y}$$

$$= \frac{\sqrt{13} \times x\sqrt{x} \times \sqrt{z}}{y} = \frac{x}{y}\sqrt{13xz}$$

4·29. A radical is said to be in its simplest form provided it contains

(i) the smallest possible index of the radical

(ii) no fraction in the radicand

(iii) no factors which are perfect n^{th} powers (in a radical of n^{th} order) as a factor of, or whole of the radicand.

Illustration : Express $\sqrt[4]{\frac{169\,x^6\,z^2}{y^4}}$ in its simplest form

$$\sqrt[4]{\frac{169\,x^6\,z^2}{y^4}} = \left(\frac{169\,x^6\,z^2}{y^4}\right)^{1/4}$$

$$= \frac{(13^2)^{1/4}\,(x^4)^{1/4}\,(x^2)^{1/4}\,(z^2)^{1/4}}{y}$$

$$= \frac{(13)^{1/2}\cdot x\cdot x^{1/2}\cdot z^{1/2}}{y}$$

$$= \frac{x}{y}\,(13\,x\,z)^{1/2}$$

$$= \frac{x}{y}\sqrt{13\,x\,z}$$

4·30. Addition and subtraction of radicals:

In addition or subtraction of radicals, each radical is first reduced in its simplest form and then terms with similar radicals are combined into a single radical.

Illustration:

$$\sqrt[8]{\frac{1}{3}} + \frac{3}{2}\sqrt{108} - \sqrt[4]{9}$$

$$= 8\left(\frac{1}{3}\right)^{\frac{1}{2}} + \frac{3}{2}\,(2^2\cdot 3^2\cdot 3)^{1/2} - (3^2)^{1/4}$$

$$= 8\,\frac{1}{3^{1/2}} + \frac{3}{2}\cdot(2^2)^{1/2}\,(3^2)^{1/2}\,(3)^{1/2} - 3^{2\cdot\frac{1}{4}}$$

$$= \frac{8}{\sqrt{3}} + \frac{3}{2}\cdot 2\cdot 3\cdot\sqrt{3} - 3^{1/2}$$

$$= \frac{8}{\sqrt{3}} + 9\sqrt{3} - \sqrt{3}$$

$$= \frac{8}{\sqrt{3}} + (9-1)\sqrt{3}$$

$$= \frac{8}{\sqrt{3}} + 8\sqrt{3}$$

$$= 8\left(\frac{1}{\sqrt{3}} + \sqrt{3}\right)$$

$$= 8\left(\frac{1+\sqrt{3}\cdot\sqrt{3}}{\sqrt{3}}\right)$$

$$= 8\left(\frac{1+3}{\sqrt{3}}\right)$$

$$= \frac{32}{\sqrt{3}}$$

$$= \frac{32\sqrt{3}}{3}.$$

4·31. Multiplication and division of radicals :

In multiplication and division of radicals, if the radicals have the same index, then the use of the laws of exponents gives the result directly. If, however, the radicals have different indices, then they must be reduced to radicals of the same index and then the laws of exponents are to be used.

Illustration 1 :

$$(\sqrt{5}+2\sqrt{3})\ (\sqrt{5}-3\sqrt{3})$$
$$=\sqrt{5}\cdot\sqrt{5}-3\sqrt{5}\sqrt{3}+2\sqrt{5}\sqrt{3}-2\cdot3\sqrt{3}\sqrt{3}$$
$$=5-3\sqrt{15}+2\sqrt{15}-6\cdot3$$
$$=-\sqrt{15}-13$$

Illustration 2 :

Divide $\sqrt[3]{4}$ by $\sqrt{8}$

$$\sqrt[3]{4} \div \sqrt{8} = 4^{1/3} \div 8^{1/2}.$$
$$= (4^2)^{1/6} \div (8^3)^{1/6} \quad [\because \text{L.C.M of 2 and 3 is 6}]$$
$$= (16)^{1/6} \div (512)^{1/6}$$

$$= \left(\frac{16}{512}\right)^{1/6} = \left(\frac{1}{32}\right)^{1/6} = \sqrt[6]{\frac{1}{32}}$$

Illustration 3.

Multiply $3\cdot\sqrt[4]{5}$ by $2\cdot\sqrt[6]{2}$

$$3\sqrt[4]{5}\times 2\sqrt[6]{2} = 3\times 5^{\frac{1}{4}}\times 2\times (2)^{\frac{1}{6}}$$

$$= 6\times (5^3)^{\frac{1}{12}}\times (2^2)^{\frac{1}{12}} \quad (\because \text{L.C.M. of 4 an 6 is 12})$$

$$= 6\times (125)^{\frac{1}{12}}\times (4)^{\frac{1}{12}}$$

$$= 6\times (125\times 4)^{\frac{1}{12}}$$

$$= 6\times (500)^{\frac{1}{12}}$$

$$= 6\ \sqrt[12]{500}$$

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4·32. Theorem :

If $a+\sqrt{b} = x+\sqrt{y}$ where a and x are rational numbers and $\sqrt{b}$ and $\sqrt{y}$ are irrational, then $a = x$ and $b = y$.

Proof :

$$a+\sqrt{b} = x+\sqrt{y}$$

$\therefore\ (a-x)+\sqrt{b} = \sqrt{y}$ · square both sides .

$$\therefore\ (a-x)^2 + b + 2\sqrt{b}\,(a-x) = y$$

$$\therefore\ 2\sqrt{b}\,(a-x) = y - b - (a-x)^2 .$$

The left hand side expression is irrational and the right hand side expression is rational. Hence, unless the coefficient of $\sqrt{b}$ on L.H.S is zero, we must have an irrational quantity equal to a rational one, which is impossible. Therefore, the coefficient of $\sqrt{b}$ must be zero.

$$\therefore\ 2(a-x) = 0$$

$$\therefore\ a = x$$

So, from $a + \sqrt{b} = x + \sqrt{y}$
we have $b = y$

Definition :

The expressions $a + \sqrt{b}$ and $a - \sqrt{b}$ is each said to be the conjugate of the other. The following three results follow directly :

(1) The sum of two conjugate surds is rational.
For, $(a + \sqrt{b}) + (a - \sqrt{b}) = 2a$ and $2a$ is rational.

(2) The product of two conjugate surds is rational.
For, $(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$ and it is rational.

(3) If $a + \sqrt{b} = x + \sqrt{y}$, then $a - \sqrt{b} = x - \sqrt{y}$. For, if $a + \sqrt{b} = x + \sqrt{y}$, then by the last theorem, $a = x$ and $b = y$; o that $a - \sqrt{b} = x - \sqrt{y}$.

4·33. To find the square root of $a + \sqrt{b}$ where $\sqrt{b}$ is a surd.

Let $\sqrt{a + \sqrt{\ }} = \sqrt{x} + \sqrt{y}$.

Squaring both sides,

$$a + \sqrt{b} = x + y + 2\sqrt{xy}$$

Now, since $\sqrt{b}$ is a surd, using the result of art. 4·32, we have

$$x + y = a \qquad \dots \ (1)$$

and $2\sqrt{xy} = \sqrt{b}$ so that $4xy = b$.

$$\therefore \ (x-y)^2 = (x + y)^2 - 4xy.$$
$$= a^2 - b.$$

$\therefore \ x - y = \sqrt{a^2 - b} \ \dots$ (2)(considering only the principal root)

Adding (1) and (2)

$$x = \frac{1}{2}\left\{a + \sqrt{a^2 - b}\right\}$$

Subtracting (2) from (1),

$$y = \frac{1}{2}\left\{a - \sqrt{a^2 - b}\right\}$$

so that $\sqrt{a+\sqrt{b}} = \sqrt{\frac{1}{2}\{a+\sqrt{a^2-b}\}} + \sqrt{\frac{1}{2}\{a-\sqrt{a^2-b}\}}.$

EXERCISE

Simplify:

1. $a^{\frac{1}{2}} \times \frac{5}{a^{7/6}} \times a^{2/3} \times (a^{1/12})^{-2}$.
2. $\sqrt{(a^{3/2}\,b^5)} \times \sqrt[4]{(b)^2\,(a)^8}$
3. $4 \cdot \sqrt[3]{72} + 5\sqrt[3]{576}$
4. $5\sqrt[3]{4} \div 2\sqrt[3]{2}$
5. $(xy^{-2}z^3)^{1/2} \times (x^3y^2z^{-3})^{1/3}$
6. $\left(\frac{125a^3c^4}{27a^6b^{-2}}\right)^{\frac{1}{3}}$
7. (i) $\frac{1}{\sqrt[5]{3x^{-10}y^{15}}} \times \frac{8y^{-16}}{11x}$

   (ii) $\sqrt{ab^2} \div \sqrt[4]{a^3b}$
8. $\left(x^{\frac{b+c}{c-a}}\right)^{\frac{1}{a-b}} \times \left(x^{\frac{c+a}{a-b}}\right)^{\frac{1}{b-c}} \times \left(x^{\frac{a+b}{c-a}}\right)^{\frac{1}{b-c}}$
9. Prove theorem 2 of Art. 4·24 by the method of induction.
10. Prove thorem 4 of Art. 4·24 by the method of induction.

    Multiply:
11. $3\sqrt{5} - 4\sqrt{2}$ by $2\sqrt{5} + 3\sqrt{2}$
12. $\sqrt{2}\,\sqrt[3]{3}$ by $\sqrt[4]{4}$
13. $\sqrt{2} + \sqrt{3}$ by $\sqrt{3} + \sqrt{5}$
14. $\sqrt{2} + 2\sqrt{3} + \sqrt{7}$ by $\sqrt{2} + 2\sqrt{3} - \sqrt{7}$

    Divide:
15. $\sqrt{a^{-5/3}\,b^3\,c^{-2/3}}$ by $\sqrt[3]{a^{1/2}\,b^4\,c^{-1}}$
16. $(24\,a^3\,b)^{1/4}$ by $\sqrt[4]{8\,ab^3}$

Simplify :

17. $\sqrt{9\,(a^2+2b)} + 2\sqrt{36\,(a^2+2b)} - 3\sqrt{16\,(a^2+2b)}$

18. $4\sqrt{8} + 5\sqrt{2} - 2\sqrt[4]{4}$

19. $\sqrt[4]{\frac{}{3}} + 3\sqrt{27} - \sqrt{3}$

20. $8\sqrt{3} - 14\sqrt{12} + 10\sqrt{2} - \sqrt{50}$

Rationalize the denominator of :

21. $\dfrac{3\sqrt{7}+1}{2\sqrt{5}-\sqrt{3}}$

22. $\dfrac{1}{x^{1/3}-y^{1/3}}$ $\left[\text{Hint : } x^3 - y^3 = (x-y)\ (x^2+xy+y^2)\right]$

23. $\dfrac{(2+\sqrt{3})}{(\sqrt{5}-\sqrt{3})}$

24. $\dfrac{\sqrt{10}+\sqrt{5}-\sqrt{3}}{\sqrt{3}+\sqrt{10}-\sqrt{5}}$

25. $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$

26. Show that,

$$\frac{1}{\sqrt{12-\sqrt{140}}} - \frac{1}{\sqrt{8-\sqrt{60}}} - \frac{2}{\sqrt{10+\sqrt{84}}} = 0.$$

27. Simplify:

$$\frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}+\sqrt{2+\sqrt{3}}} - \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-\sqrt{2+\sqrt{3}}}$$

Find the square root of each of the following:

28. $28 - 5\sqrt{12}$

29. $17 - 12\sqrt{2}$

30. $\dfrac{2+\sqrt{3}}{2}$.

ANSWERS

1. $\frac{5}{6\sqrt{a}}$
2. $a^{\frac{11}{4}} b^3$
3. $480\sqrt[3]{3}$
4. $\frac{5}{2}\sqrt[3]{2}$
5. $\frac{x^{\frac{3}{2}}\ z^{\frac{1}{2}}}{y^{\frac{1}{3}}}$
6. $\frac{5c^6}{3a}$
7. (i) $\frac{8x}{11\sqrt[5]{\ }^{3}\ y^{13}}$

   (ii) $\sqrt{\frac{b}{a}}$
8. 1
11. $6+\sqrt{10}$
12. $2\sqrt[3]{3}$
13. $3+\sqrt{6}+\sqrt{15}+\sqrt{10}$
14. $7+4\sqrt{6}$
15. $\frac{\sqrt[6]{b}}{a}$
16. $\frac{\sqrt[4]{3a^2 b^2}}{b}$
17. $3\sqrt{a^2+2b}$
18. $11\sqrt{2}$.
19. $\frac{32\sqrt{3}}{3}$

20. $5\sqrt{2}-20\sqrt{3}$

21. $\frac{1}{11}\left(6\sqrt{35}+2\sqrt{5}+3\sqrt{21}+\sqrt{3}\right)$

22. $\frac{x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}}{x-y}$

23. $\frac{2\sqrt{5}+\sqrt{15}+2\sqrt{3}+3}{2}$

24. $\frac{3\sqrt{30}+5\sqrt{15}-12-10\sqrt{2}}{7}$

25. $\frac{2\sqrt{3}+3\sqrt{2}-\sqrt{30}}{12}$

27. $\sqrt{2}+\sqrt{\ }-\frac{4}{3}\sqrt{3}$.

28. $5-\sqrt{3}$

29. $3-2\sqrt{2}$

30. $\frac{\sqrt{3}+1}{2}$

CHAPTER FIVE

Functions

5·1. In chapter 2, art. 2·22 the idea of ordered pairs has been introduced and discussed to some extent. We shall now see how this notion of ordered pairs is used to combine two sets A and B resulting in the formation of a particular type of set called the Cartesian product set.

Consider the two sets A = { 2, 5, 9 } and B = { 3,4 }. Let us form ordered pairs, with the order that the first component should be an element of A and the second component should be an element of B. We can form the pairs as follows :

$$\begin{tikzcd}
& 3 \arrow[r] & (2, 3) & & & 3 \arrow[r] & (5, 3) \\
2 \arrow[ur] \arrow[dr] & & & & 5 \arrow[ur] \arrow[dr] & & \\
& 4 \arrow[r] & (2, 4) & & & 4 \arrow[r] & (5, 4)
\end{tikzcd}$$

$$\begin{tikzcd}
& 3 \arrow[r] & (9, 3) \\
9 \arrow[ur] \arrow[dr] & & \\
& 4 \arrow[r] & (9, 4)
\end{tikzcd}$$

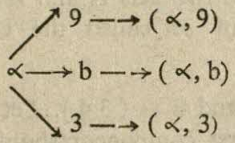
Thus, since there are two elements in B, two ordered pairs each are formed by associating an element of A with every element of B. This being done with each of the three elements of A, the total number of ordered pairs is $3 \times 2 = 6$ (Compare with the multiplication principle chapter 4,). Since an ordered pair can be looked upon as an entity or element it is possible to think of a set with these six pairs as elements. The set so formed is denoted by $A \times B$ (read A cross B) and is called the Cartesian product set of A and B. In this example therefore,

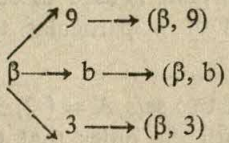
$$A \times B = \{(2, 3), (2, 4), (5, 3), (5, 4), (9, 3), (9,4)\}$$

The number of elements in this set being 6, we have

$$n(A \times B) = n(A) \times n(B).$$

Take another example. If $C = \{\propto, \beta\}$ and $D = \{9, b, 3\}$ then the ordered pairs with similar order as above, namely that "the first component is an element of C and the second component is an element of D" are formed as following :





Thus $C \times D = \{(\propto, 9), (\propto, b), (\propto, 3), (\beta, 9) (\beta, b), (\beta, 3)\}$

Note that the number of elements (i.e ordered pairs) in $C \times D$ is equal to $2 \times 3 = 6$, that is, the product of the number of elements in A and B. $\therefore$ $n(C \times D) = n(C).\ n(D)$.

Definition : For any two sets A and B, the set of all ordered pairs (x, y) where $x \in A$ and $y \in B$, is called the Cartesian product set (or simply, the product set) of A and B and is denoted by $A \times B$.

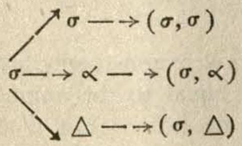
$$\text{So } A \times B = \{(x, y) / x \in A \text{ and } y \in B\}.$$

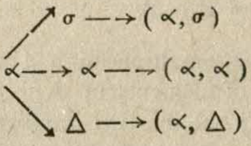
If the number of elements in A and B are m and p respectively, hen by the multiplication principle, the number of elements in $A \times B$ will be mp.

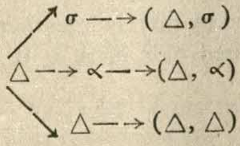
$$\text{i.e. } n(A \times B) = n(A).\ n(B).$$

5·2. The Cartesian product is a binary operation. Just as union ($\cup$) and intersection ($\cap$) in sets are operations which combine any two sets to produce a resultant set, the product ($\times$) as defined above is an operation that combines any two sets to give out a resultant set, its elements being ordered pairs.

If the sets A and B are equal, each denoted by A, then the Cartesian product set $A \times A$ is formed by pairing off each element of A with every element of itself. e. g. Let $A = \{\sigma, \propto, \triangle\}$ then, all possible ordered pairs for $A \times A$ are obtained as,







$$\therefore \quad A \times A = \{(\sigma, \sigma), (\sigma, \propto), (\sigma, \triangle), (\propto, \sigma), (\propto, \propto), (\propto, \triangle)\\ (\triangle, \sigma), (\triangle, \propto), (\triangle, \triangle)\}$$

The number of elements being $3 \cdot 3 = 9$. In general, for any set A,

$A \times A = \{(x, y) \mid x \in A, y \in A\}$

5·3 The cartesian product is non-commutative. Consider the sets

$A = \{\propto, \beta\}$ and $B = \{3, 2, 0\}$

$A \times B = \{(\propto, 3), (\propto, 2), (\propto, 0), (\beta, 3), (\beta, 2), (\beta, 0)\}$

and $B \times A = \{(3, \propto), (3, \beta), (2, \propto), (2, \beta), (0, \propto), (0, \beta)\}$

$= \{(3, \propto), (2, \propto), (0, \propto), (3, \beta), (2, \beta), (0, \beta)\}$

So that $A \times B \neq B \times A$

Note that the number of elements in A × B

$= $ (number of elements in A) (number of elements in B)

$= 2 \cdot 3 = 6.$

and, number of elements in B × A

$= $ (number of elements in B) (number of elements in A)

$= 3 \cdot 2 = 6.$

In general, $n(A \times B) = n(A).\ n(B)$ and

$$n(B \times A) = n(B).\ n(A)$$

$$\therefore n(A \times B) = n(B \times A) \text{ although}$$

$$A \times B \neq B \times A$$

(Will it be so if ordinary multiplication in integers were non-commutative?) Thus, the number of elements in A × B is equal to the number of elements in B × A, but the two product sets do not contain identical elements and therefore are unequal themselves.

Note that the set B × A could have been written by merely interchanging the positions of the components in each ordered pair of A × B. This is so because

$$A \times B = \{(x,\ y)\ /\ x \in A, \text{ and } y \in B,\}$$

$$\text{and } B \times A = \{(y,\ x)\ /\ y \in B \text{ and } x \in A\}$$

If however, A = B then A × B and B × A is each equal to A × A and hence are equal. In fact, A × B = B × A if and only if A = B.

5·4. Graph of Cartesian Product Set :

In chapter 2, we have seen that in a plane can be fixed a reference frame (preferably, though not necessarily, the rectangular Cartesian frame) by choosing a fixed pt. O as origin and two fixed lines through O called the axis of x (the horizontal line) and the axis of y (the vertical line) as the axes of reference. Any ordered pair of real numbers can then be represented as a point in the plane and conversely, each point in the plane corresponds to one and only one ordered pair of real numbers.

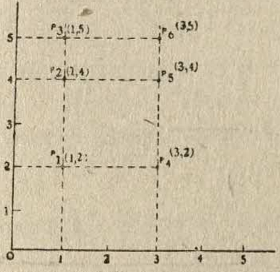
Since a Cartesian product set A × B is a set of ordered pairs and since each ordered pair can be represented geometrically by a point in the

Cartesian plane, the set A × B itself can be geometrically reprsented as a set of points.

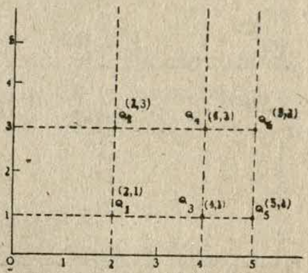
Example 1. Let A = { 1, 3 }, B = { 2, 4, 5 }

Then A × B = { (1, 2), (1, 4), (1, 5), (3, 2), (3, 4), (3, 5) }.

Take a fixed point O as origin and two fixed perpendicular lines through O as the axes, the horizontal line being the axis of x and the vertical line being the axis of y. Represent on each line the numbers



A×B (i)



B×A (ii)

1, 2, 3.... by successive points at equal intervals. Then the ordered pair (1, 2) is represented as in the figure (i), by the point P_1 and similarly each ordered paire is represented by one and only one point in the plane. The geometrical representation of the set A × B is therefore the set of points P_1, P_2, P_3, P_4, P_5, P_6, and appears pictorially as in the figure (i) shown above. The set of points { P_1, P_2, P_3, P_4, P_5, P_6 }

is called the graph of the Cartesian product set A × B.

The graph of B × A can be drawn in an exactly similar manner and in this example, it looks as in figure (ii).

Since B × A = {(2, 1), (2, 3), (4, 1), (4, 3), (5, 1), (5, 3)},

the graph of B × A is the set of points Q_1, Q_2, Q_3, Q_4, Q_5, Q_6.

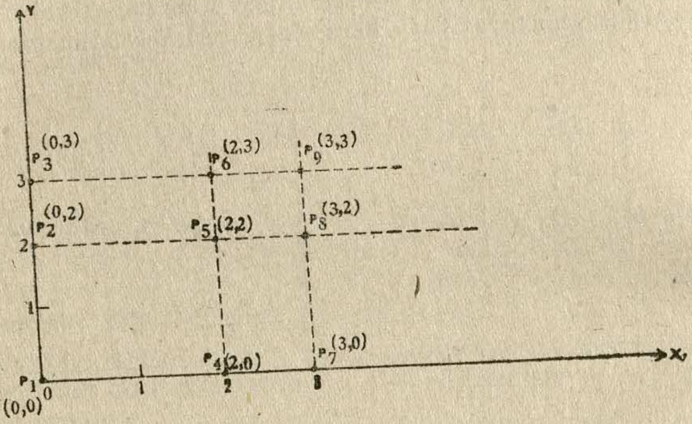
Note that the graph of A × B looks quite different from the graph of B×A though the number of points in each graph is the same. This is to be expected since $A \times B \neq B \times A$, although

$$n(A \times B) = n(B \times A)$$

Example 2. Let $A = \{0, 2, 3\}$

Then $A \times A = \{ (0, 0), (0, 2), (0, 3)\ (2, 0), (2, 2)\ (2, 3), (3, 0), (3, 2), (3, 3)\}$.

So that the graph of A × A can be plotted thus :



The graph of A × A is the set $\{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9\}$. It contains nine points as its elements.

In the examples above, notice that in each case the horizontal line is associated with that set from which the first components are selected and the vertical line is associated with that set from which the second components are chosen.

5·5. Correspondences and Functions :

The idea of an association between the elements of two sets is of fundamental significance in every branch of mathematics. If the elements of two sets can be brought under some matching (also called association or correspondence) the scope for the comparative study of many of the properties of the sets is widened considerably. For two such sets, knowing the properties of one of them many of the properties of the other can be deduced. Before discussing the various types of, and restrictions on, different associations between the elements of two given sets, let us consider a few examples.

Example 1. Let $A = \{\text{Copper, Silver, Gold, Iron, Antimony}\}$
$B = \{C_u, A_g, A_u, F_e, S_b\}$

The elements in the set B are the chemical symbols for the corresponding metals which constitute the elements of the set A. The matching or association between the elements of these sets is

$$\begin{array}{ccccc} \text{Copper} & \text{Silver} & \text{Gold} & \text{Iron} & \text{Antimony} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \text{Cu} & \text{Ag} & \text{Au} & \text{Fe} & \text{Sb} \end{array}$$

provided the matching is done with the idea "Is the chemical symbol of" in mind. Here, one metal corresponds to one and only one symbol and conversely. Such an association is called a "one to one correspondence."

Example 2. Suppose you are given the words :
good, virtue, long, beautiful,

and you are asked to write the opposite of each.

Your answer will be

bad, vice, short, ugly.

Thus the question set

A = { good, virtue, long, beautiful } gives rise to the answer set
B = { bad, vice, short, ugly }.

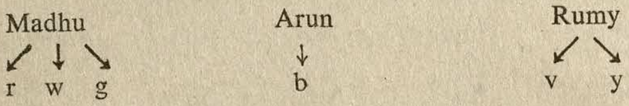
Each word in A has one and only one opposite which is an element of B. The elements of A can thus be corresponded to the elements of B with the idea "is the opposite of" in mind and the correspondence is again one to one.

Example 3 : Let A be the set of three boys Madhu, Arun, Rumy and suppose Madhu possesses three kites, red, white and green (denote them by r, w, g), Arun possesses one kite coloured blue (denote it by b) and Rumy possesses two kites coloured violet and yellow (denote by v, y). Let B be the set of these kites.

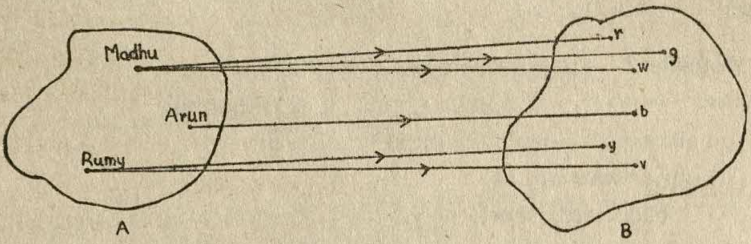
Thus A = {Madhu, Arun, Rumy}

B = { r, w, g, b, v, y }

The elements of A may be matched with the elements of B with the idea that a boy is associated to the kites he possesses. The correspondence will be as shown below :



It may also be shown as



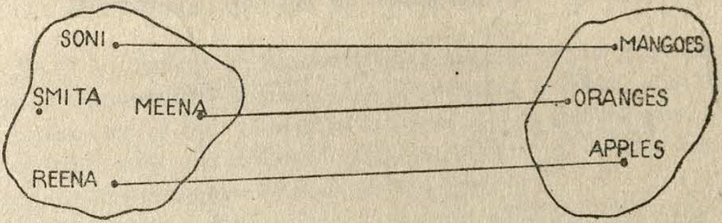
In this correspondence, an element of the set A is matched with one or more elements of the set B.

Example 4 : Let A be the set of girls Soni, Reena, Smita, Meena and B be the set of mangoes, apples, and oranges.

Thus, A = { Soni, Reena, Smita, Meena }

B = { mangoes, apples, oranges }

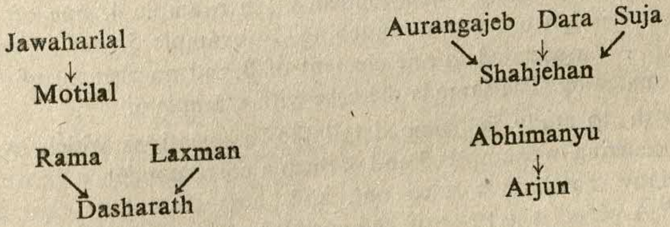
Suppose Soni likes mangoes, Meena likes oranges, Reena likes apples and Smita likes none of these fruits. If the matching between the elements of the two sets is made so as to associate a girl with the fruit she likes, then the correspondence may be shown as below :



In this association one of the elements of A (namely Smita) is left out of the correpondence.

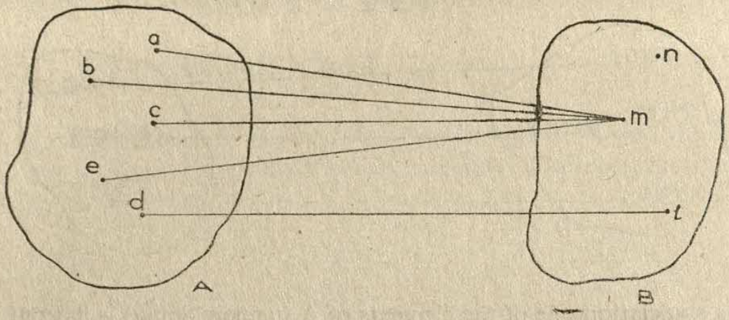
Example 5. Let A = {Jawaharlal, Aurangajeb, Dara, Suja, Rama, Laxman, Abhimanyu}
and B = {Shahajehan, Motilal, Dashrath, Arjun}

Then with the idea "an element of B is the father of an element (may be more than one) of A", the matching may be brought about in the following manner :



In this correspondence, more than one element of A may correspond to the same element of B, but not so conversely for a son cannot have more than one father although a father can have more than one son. Thus here the corespondence is many—one from A to B but not so from B to A.

Example 6 : Consider the sets A = { a, b, c, d, e }
and B = { l, m, n, } and the matching between the elements of the two sets as shown by the following figure :



This again is a many one correspondence from A to B and can be stated as "each element of A is associated to one and only one element of B". Here also, the converse of the rule of statement is not true for all elements of B.

Each of the examples given above is an illustration of a correspondence. In examples 1 and 2 the correspondence is such that each element of A is matched with one and only one element of B and conversely. In example 3 each element of A is associated to one or more elements of B. This is called a one many correspondence. In example 4, one of the elements of A is left out of the matching. In example 5, one or more elements of A is associated to one element of B and no element of A is left out of matching. The same is the case with example 6.

We wish to study in some detail the associations which avoid the cases occurring in examples 3 and 4. Such a correspondence, in which any one many matching is ruled out, and there does not exist any element of A which is left out of the correspondence, is called a "function" or a "mapping". We state the same notion in different words to make the following definition.

Definition :—If with each element of a set A is associated under a rule f one and only one element of another set B then this correspondence or rule f is called a function of A to B. It is written as f : A → B (read "f is a mapping or a function of A to B or A into B").

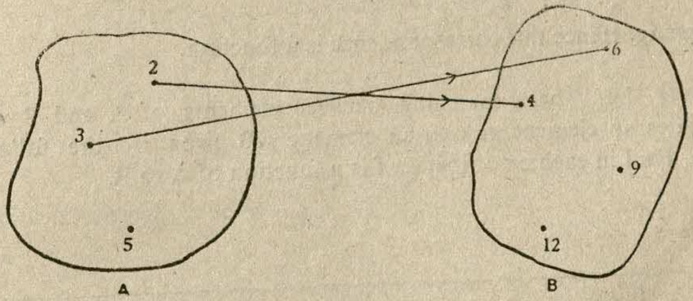
Note that (i) A function is a one way pairing process. An element of B may or may not be associated to any element of A. But each element of A must be associated to some element of B. For example, see example 6 above. The element $n \in B$ has no association.

(ii) If an element of B does have an association, it may be associated to one or more elements of A. In example 6 above, the element m is associated to four elementsa, b, c, and e of A and the element l is associated to only one element d. (iii) One element of A can not have two (or more) elements of B associated to it.

Verify that the examples 1, 2, 5, 6 given above are functions and also that the examples 3, 4 are not.

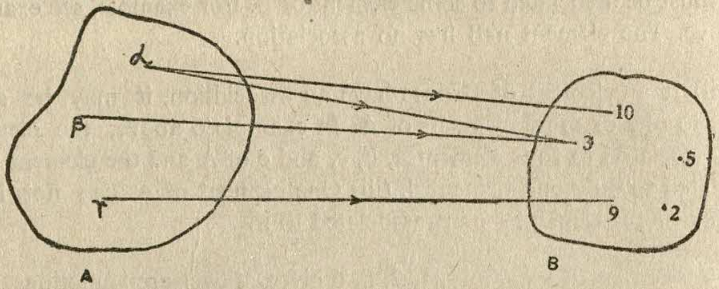
Illustrations :

(i) Let $A = \{2, 3, 5\}$, $B = \{4, 6, 9, 12\}$ and f be a rule which associates an element of A to a number which is double of that element. Then 2 in A will be associated to 4 in B, 3 in A will be associated to 6 in B, 5 in A should be associated to 10, but 10 is not in B. Therefore 5 has no association under f in B. Hence f is not a function. Pictorially, it can be seen that under the rule f, one element of A (namely 5) is

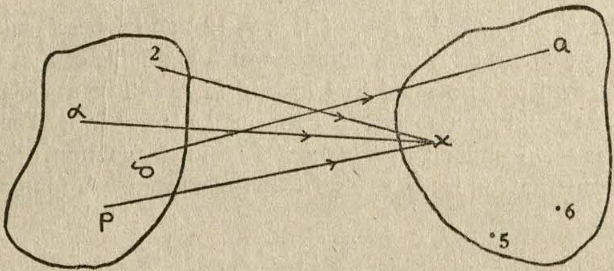


left out. For a mapping, each element of A should correspond to an element of B. Hence f is not a mapping.

(2) The following diagramis not a mapping. For, the element $\propto$ of A is associated to two elements of B which is not allowed in mapping.



(3) In this diagram, the matching satisfies all the conditions for a mapping. Hence this correspondence is a function.

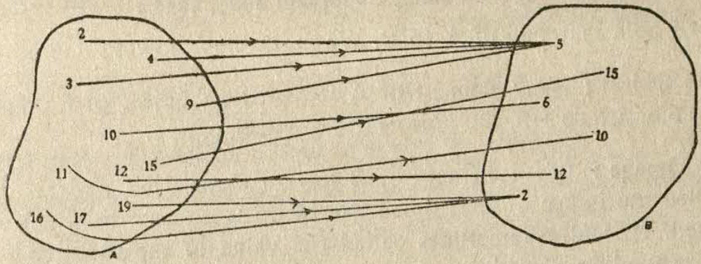


(4) Let f be a matching between elements of A and B which associates an element x∈A to an element y∈B given by the following tables. Find in each case whether f is a function of A to B.

(i)

x	2	3	4	9	10	11	12	15	16	17	19
y	5	5	5	5	6	10	12	15	2	2	2

This is a function. Pictorially it can be represented as :



Every element of A has one and not more than one associate from B.

(ii)

x	2	1	7	7	8	5	6	4
Y	2	1	5	9	α	β	γ	δ

This is not a mapping. State why.

(iii)

x	5	9	0	2	4
Y	7	7	7	7	7

This is a mapping. Verify.

(iv)

x	7	7	7	7	7
Y	5	9	0	2	4

This is not a function of A to B. Why?

5·6. Object, Image, Domain, Codomain and Range :

Let f be a mapping of A to B. We make the following definitions :

(1) **Object :** Any element of A is called an object of f. Thus in example 6 of article 5·5, the objects of f are a, b, c, d, e.

(2) **Image :** If an element of B has an association under f, then it is called the image of that object (or objects) under f to which it is associated. It is also sometimes called the value of the function f for the corresponding object.

If x is an element of A then its value or image under f is denoted by f(x).

Thus, in the last article,

In example 1.

Cu is image of copper
Ag is image of silver
Au is image of Gold
Fe is image of Iron
Sb is image of Antimony

} under the function or rule "is the chemical symbol of".

We may also write f(copper) = Cu., f(silver) = Ag, etc., where f is the function from A to B.

In example 5,

Motilal is the image of Jawaharlal
Shahjehan is the image of { Dara, Suja, Aurangajeb
Dashrath is the image of { Rama, Laxman
Arjun is the image of Abhimanyu

} under the function "is the father of".

If f : A→B, we may also write f (Jawaharlal) = Motilal, f (Dara) = Shahajehan etc.

In example 6, if f is the function of A to B, as shown in the diagram, the images under f are :

$$m \text{ is the image of } \left\{\begin{array}{ll} a & \\ b & ;\ \text{we also write it as} \\ c & f(a) = m;\ f(b) = m, \\ e & f(c) = m,\ f(e) = m. \end{array}\right.$$

l is the image of d which is also written as l $= f(d)$.

Write down the images of the corresponding objects in example 2.

(3) **Domain** : If $f : A \to B$ then the set A is called the domain of the function f.

In art 5·5, ex. 6, the domain is { a, b, c, d, e }. Find the domains in the other examples. Domain of a function f is generally denoted by D(f).

(4) **Codomain** : If $f : A \to B$ then the set B is called the codomain of f.

(5) **Range :** The set of all the images under f is called the range of the function f . The range of f is generally denoted by R(f). It is also sometimes denoted by f(A).

In art 5·5, the range in ex(1) is {Cu, Ag, Fe, Sb}; in ex (6), the range is { m, l }. What are the ranges in exs 2 and 5?

Note that for any function $f : A \to B$, $f(A) \subseteq B$.

In the concept of a function, there are three important parts: the domain (or the set of x values), the range (or the set of y values) and the rule or correspondence which associates the elements x with the corresponding elements y.

A function, therefore, is completely known when these three things are known. In practice, however, it is found that it is sufficient if we know the domain and the rule of association. For, the image of each object (and hence the range) can be obtained from the given rule. The domain of definition and the rule of correspondence must be specified in each case. Throughout the discussion that follows, unless otherwise stated, we shall always assume that the domain of definition is R, the set of real numbers.

The rule of correspondence for a function f may be denoted in various ways.

e. g. (i) $f : x \to 2x-3$ means the image of the element x is the element $2x-3$, so that image of 2 is $2 \cdot 2 - 3 = 1$. i.e, $f(2) = 1$
Similarly, $f(5) = 2 \cdot 5 - 3 = 7$, $f(-1) = 2(-1) - 3 = -5$. Such a representation may also be termed as : "the function f takes x to $2x - 3$".

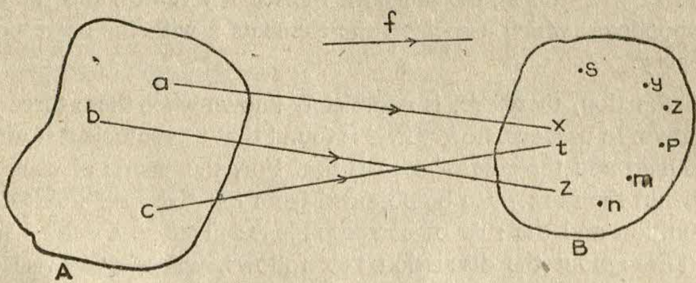
(ii) $f(x) = 2x$. This means, the image of the element x is given by 2x, so that image of 5 is $2 \cdot 5 = 10$, i.e. $f(5) = 10$; image of -4 is $2(-4) = -8$. i.e. $f(-4) = -8$ and so on.

(iii) Instead of f(x), sometimes the symbol y is used to denote the image. Thus $f : I^+ \to I$; $y = x - 5$ means f is a function from the set of +ve integers to the set of all integers given by the rule $y = x-5$, that is, $f(x) = x-5$. In such a representation, x always stands for the general object, and y for the corresponding image.

5·7. Special kinds of functions :

(i) **One—One mapping :** Let $f : A \to B$; also let the range of f be denoted by f(A). f is said to be a one-one mapping if and only if an element of f(A) is the image of only one element of A.

Example 1 : Let $f : A \to B$, the rule of association being given by the following diagram.



Here, each element in the range { x, t, z } is the image of only one object. Hence this is a one-one mapping.

Example 2. Let $f : A \to B$, where $A = \{2, 3, 4\}$ and

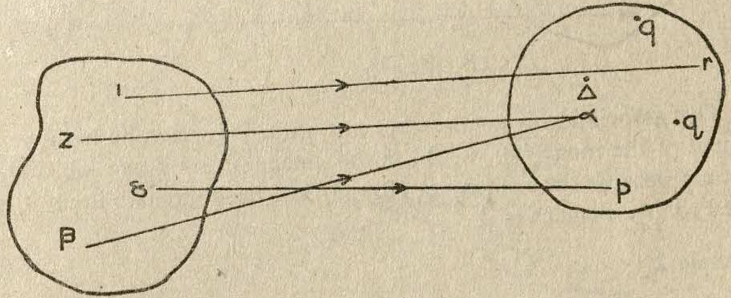
$B = \{5, 6, 7, 9, 11, 12, 15, 16\}$. Also let the rule of correspondence be $f(x) = 3x$, $x \in A$.

Then, $f(2) = 6$, $f(3) = 9$, $f(4) = 12$, so that the range is $\{6, 9, 12\}$. Each element in this range is the image of only one object.

Therefore f is a one-one mapping.

Example 3. Let $f : I^+ \rightarrow I^+$, the rule of correspondence being $y = 2x$. In other words, $f : x \rightarrow 2x$, $x \in I^+$. This is a one-one mapping. For any two different objects x_1 and x_2 ($x_1 \in I^+$, $x_2 \in I^+$) will have as their images, $2x_1$ and $2x_2$ respectively which will not be equal.

Example 4 : Let $f : A \rightarrow B$ and the association be given by the following diagram. Then the range is $\{r, \propto, p, \}$. The elements r and p are the images of one object each, but the element $\propto$ is the image of two objects 2 and β. Therefore the correspondence f is not one-one. In article 5·5, exs. 1, 2, are examples of one-one mappings. The functions defined in exs. 5, 6 are not one-one. Why?



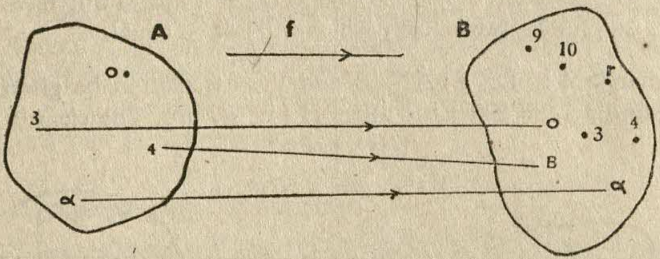
(2) **Onto mapping :** Let $f : A \rightarrow B$. Then f is called an onto mapping if and only if each element of B occurs as the image of at least one element of A.

So, if $f : A \rightarrow B$ is an onto mapping, then each element of B is in the range. Therefore the range is the whole set B. That is, $f(A) = B$. In art. 5·5, exs. 1, 2, are examples of onto mappings (in addition to being one-one); ex. 5 is onto (though not one-one) and ex. 6 is not onto (not one-one either).

If a mapping is not onto, then there should be some element of the codomain which is not in the range, that is, which does not occur as the image of any object.

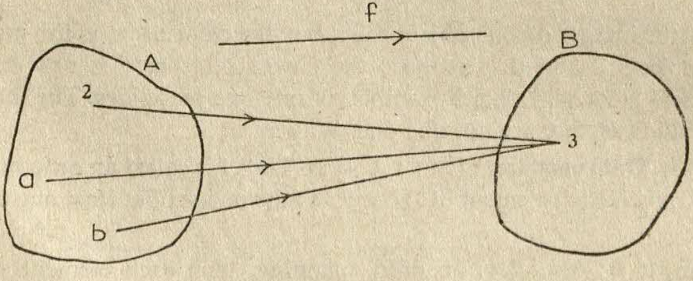
A mapping may be one-one or onto, neither or both. Given below are some illustrative examples with pictorial representations. In each of them, f : A → B.

Example 1 :



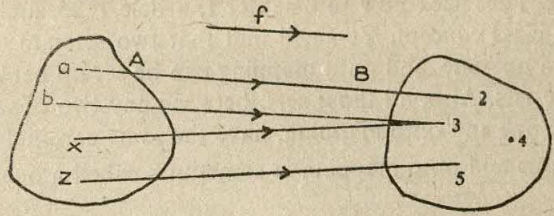
This association is one-one but not onto. It is one-one because each element of the range { 0, ∝, β } is the image of only one object. It is not onto because the elements 9, 10, r, 4, 3 of the codomain are not associated to any element of A.

Example 2 :



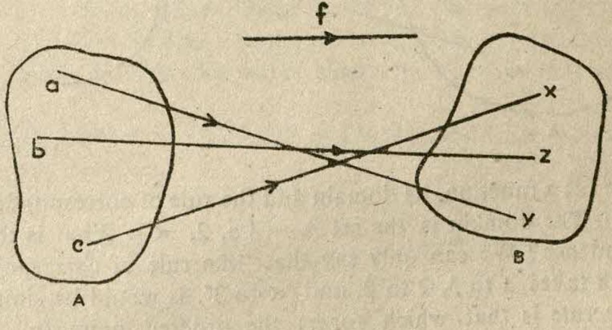
This association is onto but not one-one. It is onto because the range { 3 } is the same as the codomain B. It is not one-one because the element 3 is the image of three objects 2, a, b.

Example 3 :



This association is neither one-one nor onto. It is not one-one because the element 3 is the image of x and b, i.e., of more than one object. It is not onto because the element 4 of the codomain is not the image of any object.

Example 4 :



This association is one-one and onto. State why.

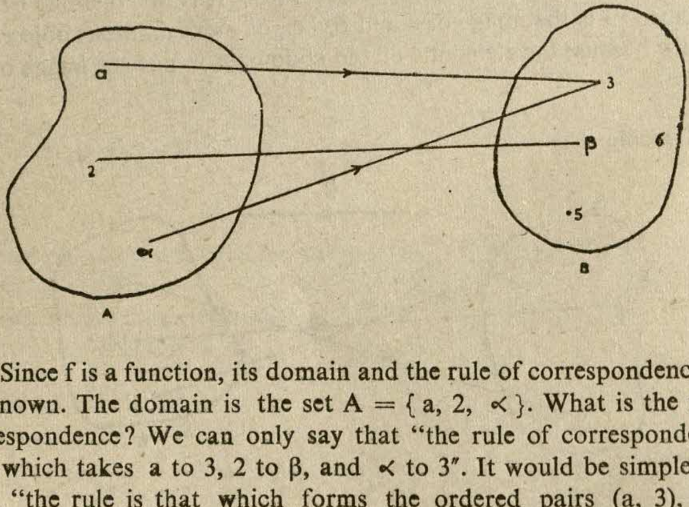
Note that the number of elements in A and B are equal, but the sets are not equal; i.e., $n(A) = n(B)$ but $A \neq B$.

Construct examples of functions like ex. 4 above which are one-one and onto. What similarity do you observe in all these functions?

Obviously, a one-one and onto mapping cannot be established between the elements of two finite sets A and B unless the number of elements in the two sets are same, that is $n(A) = n(B)$. This fact, namely that "a one-one and onto mapping existing between two sets means the number of elements in them are equal," is very important in the process of matching. Turn back now to Chapter I, article 1·25, and re-read the equivalence class concept. You will find that two sets are equivalent if and only if a one-one and onto mapping can be set up between the elements of the sets. Also, all those sets, between any two of which can be set up a one-one and onto mapping, have the same cardinal number.

A one-one and onto mapping is sometimes called a one-one correspondence.

5·8. Consider f : A → B. given by the following diagram :

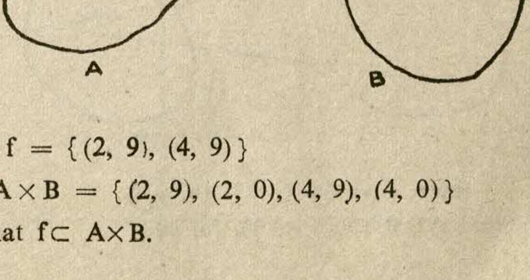


Since f is a function, its domain and the rule of correspondence must be known. The domain is the set A = { a, 2, ∝ }. What is the rule of correspondence? We can only say that "the rule of correspondence is that which takes a to 3, 2 to β, and ∝ to 3". It would be simpler if we said, "the rule is that which forms the ordered pairs (a, 3), (2, β), (∝, 3)". These three ordered pairs completely define the function. Given the function f in terms of these ordered pairs, a separate mention need not be made about its domain. In other words, we can look upon the function f as made up of these three ordered pairs. The function f, therefore, can be said to be the set of these ordered pairs, so that f = { (a, 3), (2, β), (∝, 3) }.

Take another example. Turn back to example 2 of article 5·5. Under the rule of association (which we now call a function) "is the opposite of", the word "virtue" has as its image the word "vice". The element "virtue" of the domain A therefore goes with the element "vice" of the range B and with none other. Under this association, therefore, "virtue" can be paired off with "vice" and we can think of the ordered pair (virtue, vice) under the order "Is the opposite of". The other ordered pairs of the same association in this example are (good, bad), (long, short), (beautiful, ugly). The rule or the function "Is the opposite of" connecting the elements of these two sets, instead of being expressed in these words, could have been expressed in terms of the ordered pairs (virtue, vice), (good, bad), (long, short), (beautiful, ugly). So, these ordered pairs themselves are sufficient to represent the association or function in this case. Thus, the function can be expressed as $f = \{ (\text{Virtue, vice}), (\text{good, bad}) (\text{long, short}), (\text{beautiful, ugly}) \}$.

In general, any function f of a set A to another set B is the set of ordered pairs formed with an element of A as the first component and an element of B as the second component under a rule of association which pairs off each element of A with one and only one element of B. It is this rule which distinguishes a function f of A to B, from the Cartesian product set $A \times B$. Actually, a function f of A to B is always a subset of $A \times B$. i.e., $f \subseteq A \times B$. This idea will be clearer by the following examples.

Example 1: Let $A = \{2, 4\}$. $B = \{0, 9\}$; and $f : A \to B$ be given by the following diagram.

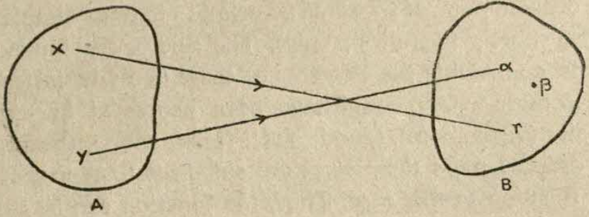


here, $f = \{(2, 9), (4, 9)\}$

and $A \times B = \{(2, 9), (2, 0), (4, 9), (4, 0)\}$

So that $f \subset A \times B$.

Example 2 : Let $A = \{x, y\}$, $B = \{\propto, \beta, r\}$; and $f : A \rightarrow B$ be given by the following diagram:



here, $f = \{(x, r), (y, \propto)\}$ whereas

$A \times B = \{(x, \propto), (x, \beta), (x, r), (y, \propto), (y, \beta) (y, r)\}$

So that $f \subset A \times B$.

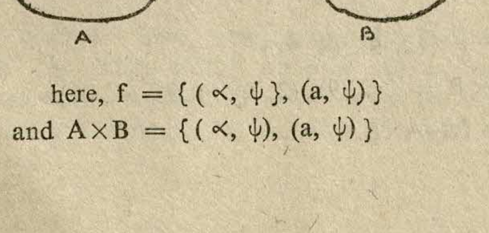
Example 3 : Let $A = \{1, 3\}$, $B = \{3, 9\}$ and $f : A \rightarrow B$ where f is given by the rule "Is the triple of" or, in other words, $f : x \rightarrow 3x$. Then the pairs formed under this rule are (1, 3), (3, 9) so that

$$f = \{(1, 3), (3, 9)\} \text{ whereas}$$

$$A \times B = \{(1, 3), (1, 9), (3, 3), (3, 9)\}$$

$$\therefore f \subset A \times B.$$

Example 4 : Let $A = \{\propto, a\}$ $B = \{\psi\}$ and $f : A \rightarrow B$ be the rule of association given by the diagram:



here, $f = \{(\propto, \psi\}, (a, \psi)\}$

and $A \times B = \{(\propto, \psi), (a, \psi)\}$

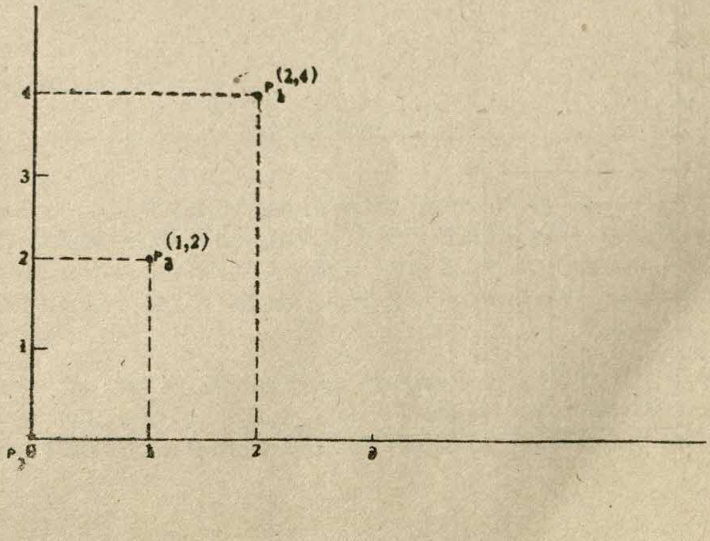
Thus $f = A \times B$.

In any case therefore, $f \subseteq A \times B$.

5·9. Graph of a function :

After realising that a function f of any set A to another set B, is nothing but a set of ordered pairs, and knowing as we do, that any ordered pair of real numbers can be interpreted geometrically under a suitable reference frame, the most natural curiosity raised is whether f can be interpreted geometrically. The answer is obviously in the affirmative, for we know that we can lay down every ordered pair geometrically as a point in the co-ordinate plane, and f being a set of ordered pairs, will be represented geometrically by the set of all such points which the ordered pairs correspond to. Let us consider a few examples.

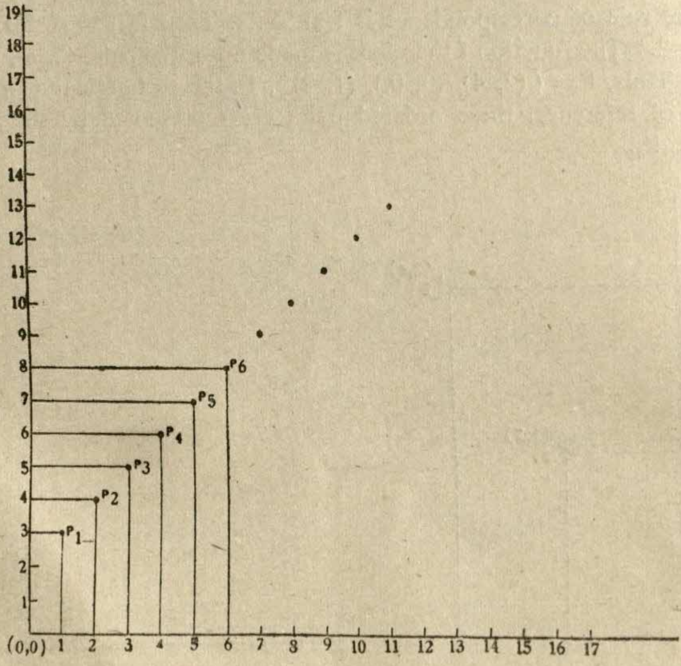
Example 1. Let $f : A \to B$ where $A = \{2, 0, 1\}$, $B = \{2, 0, 4\}$ and the rule of correspondence is $f(x) = 2x$. Then $f(2) = 4$, $f(0) = 0$, $f(1) = 2$. The function f therefore is made up of the pairs (2, 4), (0, 0), (1, 2). Thus, $f = \{(2, 4), (0, 0,), (1, 2)\}$. In the rectangular Cartesian frame of reference, these ordered pairs are represented by the points P_1, P_2, P_3.



The graph of f is therefore the set of points $\{P_1, P_2, P_3.\}$ Draw the graph f the product set $A \times B$ and observe pictorially that $f \subseteq A \times B$.

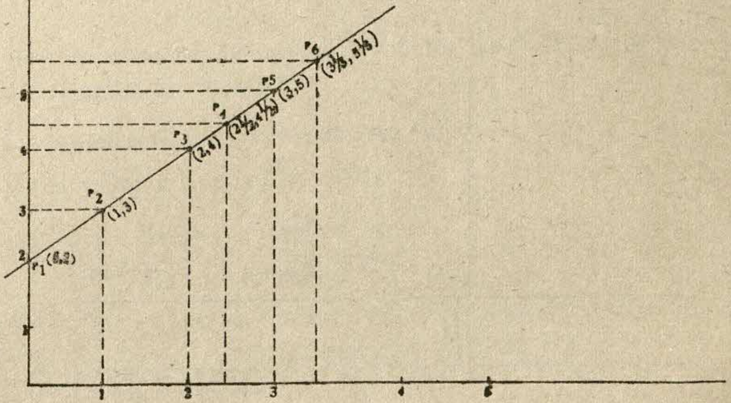
Example 2. Let $f : x \to x+2, x \in I^+$. Here, the domain is the set of all +ve integers, and the rule of correspondence is $f(x) = x + 2$. Since there are an infinite number of elements in I^+, we cannot possibly lay down all the ordered pairs forming f. We shall, however represent a fairly good number of ordered pairs so as to be able to form a general dea of what f will look like when interpreted geometrically.

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
f(x)	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19



The graph of f is the set of points P_1, P_2 .... Since there is an infinite number of points, we may visualise the assemblage of all these points as a closely knit pattern resembling a straight line. The graph is not however a complete straight line, for a line is a continuous curve whereas the set obtained here is made up of points such that there exists a gap between any two consecutive points.

Example 3. Let $f : x \to x + 2$, $x \in R$. Here too the domain is a set of infinite number of elements. Therefore, it is not possible for us to lay down geometrically all the ordered pairs constituting the function. What is more, since every real number is an object, there will be an infinite number of points in the plane very near to each other, so near that we can visualise the graph of f as a continuous curve, which, in this case, is a straght line as shown in the figure. P_1, P_2, P_3........are only some of the points of the graph of f. Note that there is no gap between the consecutive points of the graph as in ex 2, for, between any two consecutive points lie an infinite number of points, each an ordered pair belonging to f.

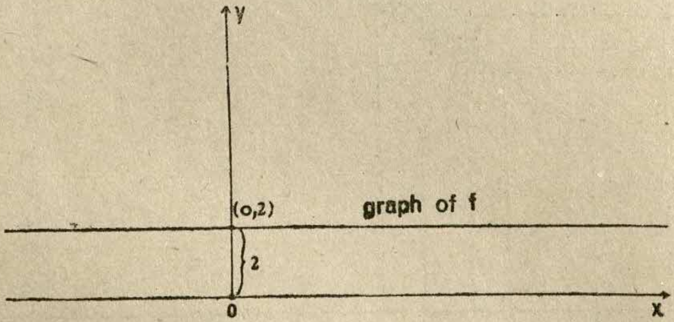


5·10. We know that given any two sets A and B, f is a function of A to B provided f is any rule whatsoever associating the elements of A with those of B, under the only restriction that f should be such that each element of A will be associated to one and only one element of B. Different rules give rise to different associations even between the same two sets. The following are a few special kinds of functions in each of which the domain of definition is the set of all real numbers.

1. **Constant function :** If f is such that $f(x) = k$, where k is a fixed real number, then f is called a constant function.

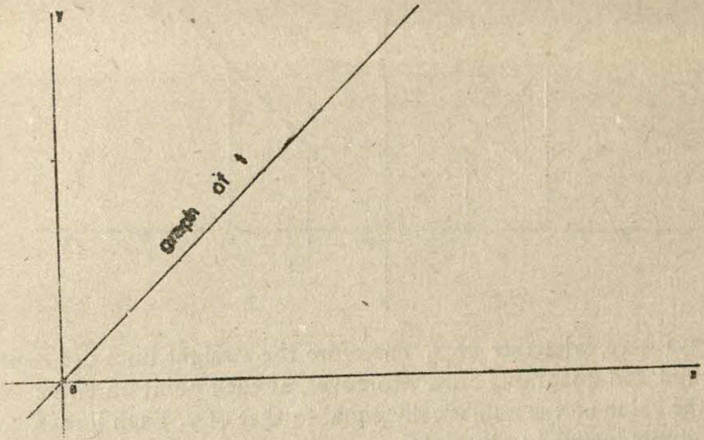
e. g. let $f(x) = 2$, $x \in R$ then $f(0) = 2$, $f(1) = 2$, $f(-5) = 2$, $f(7) = 2$ and so on.

Thus f is the set of ordered pairs (0, 2), (1, 2), (2, 2), (3, 2) etc. Verify that the graph is a straight line parallel to x axis at a distance 2 units from it on the positive side.



In general, graph of a constant function $f(x) = k$ is always a straight line parallel to x axis at a distance k from it.

2. **Identity function :** If f is defined by $f(x) = x$ then it is called an identity function. For this function, $f(0) = 0$, $f(1) = 1$ $f(2) = 2$, $f(-7) = -7$ etc. The rule can be expressed also as $y = x$ and the graph is a straight line passing through the origin, inclined equally to the two



axes (for the distances of any point on the graph from the two axes which are x and y, are the same).

3. **Modulus function** (the absolute value function):

If the defining rule is $f(x) = |x|$

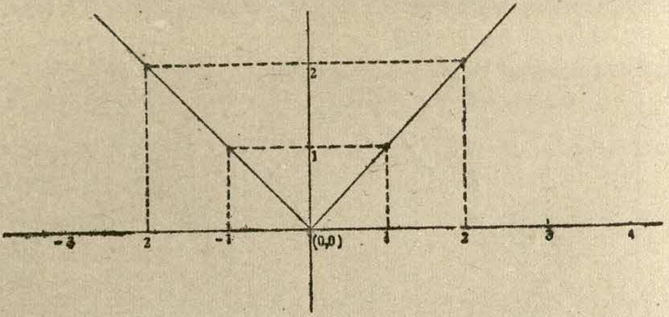
$$\text{i. e} \begin{cases} f(x) = x \text{ when } x \geqslant 0 \\ \text{and } f(x) = -x \text{ when } x < 0 \end{cases}$$

then f is called a modulus function. So, if f is a modulus function, then $f(2) = 2$, $f(-2) = 2$ $\therefore$ $(2, 2)$ and also $(-2, 2)$ are in f. $f(5) = 5$; $f(-5) = 5$ $\therefore$ $(5, 5)$ and also $(-5, 5)$ are in f.

$$f\left(\frac{8}{3}\right) = \frac{8}{3} \; ; f\left(\frac{-8}{3}\right) = \frac{8}{3} \therefore \left(\frac{8}{3}, \frac{8}{3}\right) \text{ and also}$$

$\left(\frac{-8}{3}, \frac{8}{3}\right)$ are in f. Note that $f(0) = 0$ so that $(0, 0) \in f$.

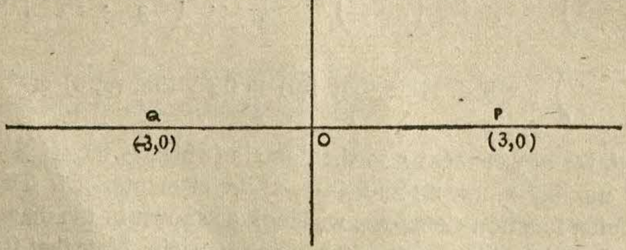
In general, for any element x in the domain (which is the set R). i.e., for any real number x, $(x, x,)$ and $(-x, x)$ are elements of f. The graph of a modulus function therefore, will look as shown in the diagram and is a pair of straight lines passing through the origin. Note that f(x) i.e., y



is always +ve, whatever be x. Therefore the straight lines are confined to 1st and 2nd quadrants only. Moreover, at each point on either of the lines, the value of x is numerically equal to that of y. Each line is, therefore, equally inclined to the axes.

5·11. Zeros of a function : Let $f : R \rightarrow R$; then any value of x for which f(x) is zero, is called a zero of the function f.

e. g. let $f : x \rightarrow x^2 - 9$, $x \varepsilon R$ then, the defining rule is $f(x) = x^2 - 9$. At $x = 3$, and $x = -3$, $f(x) = 0$. That is, $f(3) = 0$ and $f(-3) = 0$. Therefore 3 and -3 are the zeros of this function f. Note that $f(3) = 0$ means $(3, 0) \varepsilon f$ and $f(-3) = 0$ means $(-3, 0) \varepsilon f$ and these ordered pairs



8. $\frac{1}{2}$, $\frac{2}{7}$, -1, $\frac{4}{25}$, -4. $g(-4)$ is undefined. Restriction : $x \neq -4$

9. $0, 2, 4\sqrt{2}, \sqrt{62}, 8$

10. $1, 7, 1, 7, \frac{7}{3}, \frac{10}{3}$

11. $1, 243, 81, \frac{1}{27}, \frac{1}{81}, \sqrt{3}, \frac{1}{\sqrt{3}}$

12. 5, 5, 5, 5, 5, 5, 5

13. A, B, $\{4, 9, 16, 25\}$

14. A, B, $\{2\propto, 2\beta, 2\triangle\}$

15, A, B, { □, ▽ }

16. A, B, $\{5\}$

17. One-one but not onto

18. One-one and onto

19. Onto but not one-one

20. Neither one-one nor onto.

5·12. Linear function :

A function f for which the defining condition is $f(x) = mx + c$ where m is a non-zero real constant and c is any real constant, is called a linear function.

The defining condition is also written as

$$y = mx + c. \quad x \in R$$

For a linear function it is necessary that the power of x in the expression for f(x) does not exceed 1.

Thus the functions

$$f : x \rightarrow 3x + 2, \quad x \in R$$

$$g : x \rightarrow x - 5, \quad x \in R$$

$$h : x \rightarrow \frac{3}{2}x + 7, \quad x \in R$$

8. If $g : x \rightarrow \frac{2}{x+4}$ find $g(o)$, $g(3)$, $g(-6)$, $g\left(8\frac{1}{2}\right)$, $g\left(-\frac{9}{2}\right)$ what do you infer about $g(-4)$? what therefore should be the restriction on the domain of g ?

9. $h : x \rightarrow 2\sqrt{x}$, find $h(0)$, $h(1)$, $h(8)$, $h\left(15\frac{1}{2}\right)$, $h(16)$.

10. If $F : x \rightarrow |2x + I|$, find $F(0)$, $F(3)$, $F(-1)$, $F(-4)$, $F\left(-\frac{5}{\ }\right)$, $F\left(\frac{7}{6}\right)$

11. If $t : x \rightarrow 3^x$, find $t(0)$, $t(5)$, $t(4)$, $t(-3)$, $t(-4)$, $t\left(\frac{1}{2}\right)$, $t\left(-\frac{1}{2}\right)$.

12. If $f : x \rightarrow 5$ find $f(9)$, $f(-1)$, $f(0)$, $f(a)$, $f(c)$, $f\left(\frac{1}{5}\right)$, $f\left(-\frac{1}{7}\right)$

Find the domain, codomain, range of the following functions $f : A \rightarrow B$.

13. rule of correspondence : $f(x) = x^2$, $x \in A$.
$A = \{2, 3, 4, 5,\}$ and $B = \{1, 2, 4, 5, 6, 9, 11, 16, 25, 27, 30\}$

14. $A = \{\alpha, B, \triangle\}$, $B = \{1, 2, 3, 2\alpha, 2\beta, 2\triangle, 10\}$
rule of correspondence being $f(x) = 2x$, $x \in A$

15. $A = \{\square, \nabla\}$, $B = \{\theta, \emptyset, 3, \square, k, \nabla\}$
rule of correspondence being $f(x) = x$, $x \in A$

16. $A = \{w, \triangle, 3\}$, $B = \{5, 9, 0, 7\}$ rule of correspondence is $f : x \rightarrow 5$

Find whether the function $f : A \rightarrow B$ is one-one, onto, one one and onto, neither one one nor onto, in each of the following examples. Draw a figure in each case.

17. $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$ rule of correspondence being $f(x) = 2x$, $x \in A$

18. $A = \{1, 2, 3\}$, $B = \{2, 6, 4\}$, rule of correspondence same as in example 17 given above.

19. $A = \{-\propto, -2, \propto, 2, \beta, 3\}$, $B = \{\propto^2, \beta^2, 4, 9\}$ rule of correspondence being $f(x) = x^2$

20. $A = \{-\propto, -2, \propto, 2, \beta, 3\}$ $B = \{\propto^2, \beta^2\ 4, 9, t\}$ rule of correspondence being same as in example 19 above.

ANSWERS

1. (i) $\{(0, 1), (0, 3), (0, 5), (2, 1), (2, 3), (2, 5)\}$
   (ii) $\{(1, 9), (3, 9), (5, 9)\}$
   (iii) $\{(0, 9), (2, 9)\}$
   (iv) $\{(9, 1), (9, 3), (9, 5)\}$
   (v) $\{(9, 0), (9, 2)\}$
   (vi) $\{(0, 0), (0, 2), (2, 0), (2, 2)\}$
   (vii) $\{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5,), (5, 1), (5, 3), (5, 5)\}$
   (viii) $\{(9, 9)\}$
   (ix) $\{(0, 1), (0, 3), (0, 5), (2, 1), (2, 3), (2, 5), (1, 9), (3, 9), (5, 9)\}$
   (x) $\{(9, 0), (9, 2), (1, 9), (3, 9), (5, 9)\}$
   (xi) ϕ

2. (i) m^2, (ii) n^2, (iii) p^2, (iv) mn, (v) np, (vi) pm, (vii) nm, (viii) pn, (ix) mp.

3. (i) True, (ii) False (iii) False (iv) True (v) False (vi) False (vii) True

5. $A \times \phi = \phi$. yes.

6. $3, -7, 28, 58, -104\frac{1}{2}, -157, 5a + 3$

7. $-9, 26, 35\ 68, -7, -\frac{88}{9}$

represent two points P and Q respectively, on the x–axis. Since P and Q lie on the graph of f, and are at the same time zeros of f, we observe that the zeros of a function are the points at which the graph of the function meets the axis of x.

EXERCISE

1. If $A = \{0, 2\}$, $B = \{1, 3, 5\}$, $C = \{9\}$ find (i) $A \times B$, (ii) $B \times C$, (iii) $A \times C$, (iv) $C \times B$, (v) $C \times A$, (vi) $A \times A$, (vii) $B \times B$, (viii) $C \times C$, (ix) $(A \times B) \cup (B \times C)$ (x) $(C \times A) \cup (B \times C)$, (xi) $(A \times B) \cap (A \times C)$

2. Let m be the number of elements in set A, n the number of elements in set B and p the number of elements in set C. Find the number of elements in (i) $A \times A$, (ii) $B \times B$, (iii) $C \times C$, (iv) $A \times B$, (v) $B \times C$, (vi) $C \times A$, (vii) $B \times A$, (viii) $C \times B$, (ix) $A \times C$.

3. Let $A = \{r, s, t\}$, $B = \{4, r\}$, $C = \{t\}$
State whether the following are true or false:
(i) $A \times C \subset A \times A$, (ii) $B \times C \subset B \times B$, (iii) $C \times C \subset B \times B$, (iv) $C \times C \subset A \times A$, (v) $C \times C \subset A \times B$, (vi) $B \times C \subset A \times B$, (vii) $C \times B \subset A \times B$

4. If $A = \{L, i, z\}$, $B = \{i\}$, $C = \{S, u, e\}$
contruct the sets (i) $A \times B$, (ii) $B \times A$, (iii) $B \times C$, (iv) $C \times B$, (v) $(A \cup B) \times B$ (vi) $B \times B$.
hence verify that $B \times B \subset A \times B$, $B \times B \subset B \times A$.
Is $B \times B \subset B \times C$ and $B \times B \subset C \times B$?

5. What is $A \times \emptyset$ for any set A, $\emptyset$ being the null set ? Is $\emptyset \subseteq A \times B$ for any two sets A and B ?

6. If $f : x \rightarrow 5x + 3$ find $f(0)$, $f(-2)$, $f(5)$, $f(11)$,
$f\left(-21\frac{1}{2}\right)$, $f(-32)$, $f(a)$.

7. If $f : x \rightarrow 2x^2 + 3x - 9$ find $f(0)$, $f(-5)$, $f(4)$,
$f(-7)$, $f\left(\frac{1}{2}\right)$, $f\left(-\frac{1}{3}\right)$

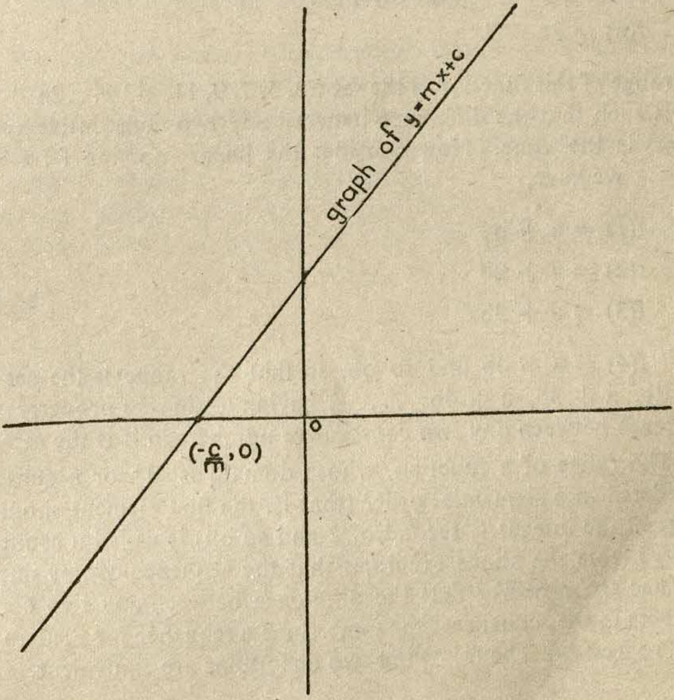
is a linear function each. The ex. 3 of art 5·9 is an example of a linear function. As in this example, the graph of every linear function is always a straight line.

To find the zeros of a linear function, we equate f(x) to zero and find for what values of x this is satisfied.

Now, $f(x) = 0$ means $mx + c = 0$

So that $x = -\frac{c}{m}$ (remember $m \neq 0$, so division by m is allowed).

$-\frac{c}{m}$ is therefore the zero of the linear function $f(x) = mx + c$. Also $f\left(-\frac{c}{m}\right) = 0$, and, $\left(-\frac{c}{m}, 0\right)$ lies on the graph of f. The



graph of f will therefore be a straight line passing through the point $\left(-\frac{c}{m}, 0\right)$

5·13. Arithmetic Progression :

Consider the linear function $f : x \rightarrow 2x + 1,\ x \in I^+$ then $f(x) = 2x + 1$ so that we have

$f(1) = 3$
$f(2) = 5$
$f(3) = 7$
$f(4) = 9$
$f(5) = 11$
$f(6) = 13$, and in general,
$f(n) = 2n + 1$

The range of this function is the set $\{3, 5, 7, 9, 11, 13 \ldots . 2n + 1 \ldots .\}$ and is such that the difference between any two consecutive numbers in the set is the same. Now consider the linear function $f : x \rightarrow a + bx$, $x \in I^+$. we have,

$f(1) = a + b$
$f(2) = a + 2b$
$f(3) = a + 3b$

$f(4) = a + 4b$ and so on, so that the range is the set $\{a + b, a + 2b, a + 3b, a + 4b \ldots \ldots,\}$ having again the property that the difference between any two consecutive numbers in it is the same.

The range of a function, whose domain is I^+ or a subset of I^+, considered in a sequential order (that is, the first element should be the image of the integer 1, second of 2 and so on) is called a sequence. We observe from the above examples that the sequence of any linear function has the property that the difference between any two consecutive numbers in the sequence is the same. Such a sequence is called an Arithmetic Progression. The following two definitions are equivalent.

Definition 1 : An arithmetic progression is any sequence of real numbers in which each term after the first is obtained by adding a fixed real number to the previous term.

Definition 2. An arithmetic progression is any sequence for which the defining function is linear.

Illustrations :

1. The finite sequence 4, 7, 10, 13, 16, is an arithmetic progression with defining function $f : x \rightarrow 3x + 1$ and domain $\{1, 2, 3, 4, 5\}$. Here, the difference between any two consecutive terms is 3. (Note that the coefficient of x in the defining function is 3.)

2. The sequence 12, 17, 22, 27, 32 is an arithmetic progression with defining function $f : x \rightarrow 5x - 3$ and domain $\{3, 4, 5, 6, 7\}$. Here, the difference between any two consecutive terms is 5, which is also the coefficient of x in the defining function.

5·14. We shall now discuss some algebraic properties of an arithmetic progression. The following notations will hereafter be used for any general arithmetic progression (written in short as A. P) given by $t_1, t_1 + d, t_1 + 2d \ldots$; t_1 = the first term, d = the common difference, n = number of terms, t_n = the nth term

The nth term : In the A. P. $t_1, t_1 + d, t_1 + 2d, \ldots\ldots$ the nth term, denoted by t_n may be found by the following methods.

Method 1 :

$$\begin{aligned}
t_2 - t_1 &= d \\
t_3 - t_2 &= d \\
t_4 - t_3 &= d \\
&\cdot\ \cdot\ \cdot\ \cdot \\
&\cdot\ \cdot\ \cdot\ \cdot \\
t_{n-1} - t_{n-2} &= d \\
t_n - t_{n-1} &= d.
\end{aligned}$$

$$\text{adding, } t_n - t_1 = d + d + d + \ldots \text{ (n-1) times}$$

$$= (n-1)d$$

$$\therefore\ t_n = t_1 + (n-1)d.$$

Method 2 : (method of induction)

We wish to establish that $t_n = t_1 + (n - 1)d$.

Let P(n) be $t_n = t_1 + (n - 1)d$. ......(i)

Then, P(1) is, $t_1 = t_1 + 0$

$\therefore$ P(1) is true ....(ii)

Let P(m) be true. $\therefore$ $t_m = t_1 + (m - 1)$ d ....(iii)

now, $t_{m+1} = t_m + d$ (Since the common difference of the A. P is d)

$= t_1 + (m - 1)d + d.$

$= t_1 + md$

$= t_1 + \{(m + 1) - 1\}$ d......(iv)

From (i), P(m + 1) is, $t_{m+1} = t_1 + \{(m + 1) - 1\}d$

$\therefore$ from (iv), P(m + 1) is true. . . .(v)

thus, from (i), (iii) and (v),

(1) P(1) is true. and

(2) If P(m) is true then P(m + 1) is true.

$\therefore$ The formula $t_n = t_1 + (n - 1)$ d is true for all positive integral values of n.

Example 1 : Find the 25th term of the series $-7, -5, -3, -1, 1, 3, 5, 7$. ....

here, $t_1 = -7$, d = the common difference

$= t_2 - t_1 = -5 - (-7) = 2.$

n = 25.

$\therefore t_{25} = -7 + (25 - 1) \cdot 2 = 41.$

Example 2 : Find the A. P. of 5 terms in which the first term is $\frac{4}{3}$ and the last term is $5\frac{1}{3}$.

here, $t_1 = \frac{4}{3}$, $n = 5$, $t_5 = 5\frac{1}{3} = \frac{16}{3}$.

$\therefore$ from the formula $tn = t_1 + (n - 1)$ d,

$\frac{16}{3} = \frac{4}{3} + (5-1)\,d \quad \therefore d = 1$

$$\therefore\ t_1 = \frac{4}{3},\ t_2 = t_1 + d = \frac{4}{3} + 1 = \frac{7}{3},\ t_3 = \frac{10}{3},$$

$$t_4 = \frac{13}{3} \text{ and } t_5 = \frac{16}{3}.$$

$$\therefore \text{ The A. P is } \frac{4}{3}, \frac{7}{3}, \frac{10}{3}, \frac{13}{3}, \frac{16}{3}.$$

5·15. To find the sum of n terms in an Arithmetic Progression.

Let the A. P. be $t_1, t_1 + d, t_1 + 2d \ldots\ \ldots t_1 + (n - 1)\,d \ldots$, and let S denote the sum of n terms of the A. P. then,

$$S = t_1 + (t_1 + d) + (t_1 + 2d) + \ldots\ldots + (t_1 + \overline{n-1}\,d)$$

also, by writing the series in the reverse order,

$$S = [\,(t_1 + (n-1)\,d\,] + [\,(t_1 + (n-2)\,d\,] + \ldots\ldots + t_2 + t_1$$

adding the corresponding terms of these two series,

$$2S = [(2t_1 + (n-1)\,d)] + [(2t_1 + (n-1)\,d)] + (2t_1 + (n-1)d] + \ldots\ldots\ldots \text{ for n terms.}$$

$$= n\,[\,2t_1 + (n-1)\,d\,]$$

$$\therefore S = \frac{n}{2}\left[2t_1 + (n-1)\,d\right] \ldots.(2)$$

Since $t_n = t_1 + (n-1)\,d$, therefore (2) may also be expressed as

$$S = \frac{n}{2}\ (t_1 + t_n) \ldots.(3)$$

Example 1 : Sum $3, \frac{7}{3}, \frac{5}{3}$ .... to 16 terms.

This is an A.P in which $t_1 = 3$,

$$n = 16,\ d = t_2 - t_1 = \frac{7}{3} - 3 = -\frac{2}{3}.$$

∴ Using the formula (2) above,

$$S = \frac{16}{2}\left[2\cdot3 + (16-1)\left(-\frac{2}{3}\right)\right]$$

$$= -32.$$

Example 2 : The sum of 15 terms of an A. P. is 600 and the first term is 5. Find the last term.

here, $t_1 = 5$, $n = 15$, $S = 600$, the last term $= t_n$. Using the formula (2),

$$600 = \frac{15}{2}(5 + t_n) \qquad \therefore t_n = 75$$

5·16. The Arithmetic Mean :

The terms of an A. P. between any two fixed terms are called the arithmetic means between these two terms.

If a, b, c, are any three numbers in A. P. then b is the arithmetic mean (A. M.) between a and c. Since the three terms are in A. P., the difference between the second and the first must be the same as the difference between the third and the second terms.

$$\therefore b-a = c-b$$

$$\therefore b = \frac{a+c}{2} \quad \ldots.(4)$$

Example 1. Find the arithmetic mean between (i) 5 and 9 (ii) $3\frac{3}{4}, \frac{1}{4}$ (iii) $-5, 6$.

Using formula (4) above, we have

$$\text{(i) A. M} = \frac{5+9}{2} = 7$$

$$\text{(ii) A. M} = \frac{3\frac{3}{4} + \frac{1}{4}}{2} = 2$$

$$\text{(iii) A. M} = \frac{-5+6}{2} = \frac{1}{2}.$$

Example 2 : Insert 5 arithmetic means between 14 and 86.

Let the A. Ms be a_1, a_2, a_3, a_4, a_2. Then on inserting them between 14 and 86, the series so formed is an A. P. that is,

14, a_1, a_2, a_3, a_4, a_5, 86, is an A. P in which $t_1 = 14$, $t_7 = 86$,

$n = 7$. using the formula of article 5·14, the common difference d can be obtained as follows.

$$86 = 14 + (7-1)\,d. \;\therefore\; d = 12.$$

$$\therefore\; a_1 = t_1 + d = 14 + 12 = 26$$

$$a_2 = a_1 + d = 26 + 12 = 38$$

$$a_3 = a_2 + d = 38 + 12 = 50$$

$$a_4 = a_3 + d = 50 + 12 = 62$$

$$a_5 = a_4 + d = 62 + 12 = 74.$$

∴ The arithmetic means are 26, 38, 50, 62, 74.

5·17 The natural numbers 1, 2, 3, 4....form an A. P. in which the first term is 1, and common difference is also 1. We proceed to find in this article, the sum of, and the sum of the squares of, the first n natural numbers.

(a) Let S_1 denote the sum of the first n natural numbers. Then

$$S_1 = 1 + 2 + 3 + \dots + n$$

$$= \frac{n}{2}\,[2\cdot 1 + (n-1)\cdot 1]$$

$$= \frac{n\,(n+1)}{2}$$

(b) Let S_2 denote the sum of the squares of first n natural numbers. That is, $S_2 = 1^2 + 2^2 + 3^2 + \dots + n^2$.

We know that for any real number n,

$$n^3 - (n-1)^3 = 3n^2 - 3n + 1.$$

Since this holds for any real value of n, putting n = 1, 2, 3 ...... etc, we obtain

$$1^3 - 0^3 = 3\cdot 1^2 - 3\cdot 1 + 1$$

$$2^3 - 1^3 = 3\cdot 2^2 - 3\cdot 2 + 1$$

$$3^3 - 2^3 = 3\cdot 3^2 - 3\cdot 3 + 1$$

and so on.

Lastly , $n^3 - (n-1)^3 = 3\cdot n^2 - 3\cdot n + 1$.

Adding, $n^3 = 3[\,1^2 + 2^2 + \,..\, + n^2] - 3\,[1 + 2 + \,..\, + n] + n$

$$= 3S_2 - 3S_1 + n$$

$$\therefore\ 3S_2 = n^3 + 3S_1 - n$$

$$= n^3 + 3\ \frac{n(n+1)}{2} - n.$$

$$= n\left\{ n^2 - 1 + \frac{3(n+1)}{2} \right\}$$

$$= \frac{n(n+1)\ (2n+1)}{2}$$

So that $S_2 = \dfrac{n(n+1)\ (2n+1)}{6}$

It is left for the students to prove that if S_3 denotes the sum of the cubes of the first n natural numbers, that is,

$$S_3 = 1^3 + 2^3 + 3^3 + \ldots\ + n^3$$

Then $S_3 = \dfrac{n^2(n+1)^2}{4}$, So that $S_3 = S_1^2$.

[Hint : Consider $n^4 - (n-1)^4 = 4\,n^3 - 6\,n^2 + 4\,n - 1$.

Put $n = 1, 2, \ldots n$ and add.]

Each of the three above formulae can be proved by the method of induction.

EXERCISE

1. (i) Find the value of r so that $8r + 4$, $6r - 2$, $2r - 7$ form an A. P. (ii) Find the 10^{th} term of the series $-3, 3, 9, 15 \ldots\ldots$
2. Find the 12^{th}, 15^{th} and 23^{rd} terms of $8, 3, -2, -7, -12 \ldots$
3. Find the m^{th} and r^{th} terms of $\frac{3}{4}, \frac{2}{3}, \frac{7}{12} \ldots\ldots$
4. Write the first four terms of an arithmetic progression in which the 5^{th} term is -5 and the 30^{th} term is -55.
5. In an A. P. the 3^{rd} term is 18 and the seventh term is 30. Find the 13^{th} term.

6. Find the sum to 15 terms of the A. P. $1, \frac{5}{4}, \frac{3}{2} \cdots$

7. Find the sum to 22 terms of the A. P. $\frac{1}{3}, -\frac{1}{6}, -\frac{2}{3}$ ....

8. In an A. P., the first term is 6, common difference is 5 and the n^{th} term is 36. Find the sum to n terms.

9. Find n if the sum to n terms of an A.P is 40 in which the first term is 4 and common difference is also 4. What is the nth term?

10. Find the sum of 10 even integers starting from and including 6.

11. (1) Find the arithmetic mean between

    (i) 7 and -15, (ii) $\frac{3}{4}$ and $\frac{4}{3}$, (iii) -4 and 0

    (2) Insert 4 arithmetic means between 3 and -12.

12. Find the sum of all odd integers between and including 9 and 35.

13. In a sack race, a boy covers 3ft. in each jump. If he is required to cover 100 ft. in how many jumps will he be 7 ft short of the stopping line?

14. How many times in all will a wall-clock strike between 1 P. M. and 10 A. M. including the chimes at these two hours?

15. If a man joins a job at Rs. 5000 a year with an increment of Rs. 20 per year then what will his annual salary be at the end of 12 years?

16, What will be the cost of a painting by Rambrandt after 10 years if it costs Rs. 30000 today and if the price goes up steadily by Rs. 200 every year?

17. A person borrows Rs. 2500 at the rate of 5% simple interest per annum. What amount will he have to pay at the end of 6 years to clear off his debt?

18. If a, b, c represent the m^{th}, n^{th} and r^{th} terms of an A. P., Prove that

$$\begin{vmatrix} a & b & c \\ m & n & r \\ 1 & 1 & 1 \end{vmatrix} = 0$$

19. Prove by method of induction the following formulae.

(i) $1^2+2^2+\ldots\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$

(ii) $1^3+2^3+\ldots\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2$

20. Sum the series $1+(1+2)+(1+2+3)+\ldots\ldots$ to n terms.

21. Find the sum of n terms of a series whose n^{th} term is $3n^2-5n+9$

22. Sum the series $(1+2-3)+(4+5-6)+(7+8-9)\ldots\ldots$ ....to 3n terms

23. Sum the series

$1\cdot4+2\cdot6+3\cdot8+4\cdot10+\ldots\ldots$ to n terms

24. Sum the series

$1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+\ldots\ldots$ to n terms.

25. Sum to n terms the series

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\frac{1}{1+2+3+4}+\ldots\ldots$$

$$\left[\text{Hint}: \ T_n=\frac{1}{\frac{n(n+1)}{2}}=2\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

26. Prove, by the method of induction, the formula for the sum to n terms of an arithmetic progression.

ANSWERS

1. (i) $\frac{1}{2}$ (ii) 51

2. $-47, -62, -102$

3, $\frac{5}{6}-\frac{m}{12}, \frac{5}{6}-\frac{r}{12}$

4. $3, 1, -1, -3$

5, 48

6. $41\frac{1}{4}$

7. $-\frac{1}{2}$

8. 147

9. 4, 16

10. 150

11. (1) (i) -4 (ii) $\frac{25}{4}$ (iii) -2

(2) $\frac{6}{5}, -\frac{4}{5}, -\frac{13}{5}, -\frac{22}{5}$

12. 308

13. 31

14. 55

15. 5240

16. 32000

17. Rs. 3250

20. $\frac{n(n+1)\,(n+2)}{6}$

21. $n^3 - n^2 + 7n$

22. $\frac{3}{2}\,n(n-1)$

23. $\frac{2}{3}\,n(n+1)\,(n+2)$

24. $\frac{n(n+1)\,(n+2)\,(n+3)}{4}$

26. $\frac{2n}{n+1}$

5·18. Linear equations :

Consider the equation $3x - 4y = 1$.

This equation is satisfied by the values $x = 3$, $y = 2$, so that (3, 2) is a solution of the given equation. There are, however, other solutions of the same equation. For instance, $\left(2, \frac{5}{4}\right)$, $(-1, -1)$, $\left(\frac{1}{3}, 0\right)$ are each a solution of the equation $3x - 4y = 1$. In fact, there are an infinite number of pairs of real numbers (x, y) which satisfy the given equation. The solution set of the equation is therefore an infinite set and can be expressed as $\{(x, y) / 3x - 4y = 1\}$

Let us interpret this equation geometrically. Since the given equation can be written as

$$y = \frac{3}{4}x - \frac{1}{4}$$

which is of the form $y = mx + c$, and as we have already seen in Art. 5·12 that any linear function $y = mx + c$, $m \neq 0$ represents a straight line, the graphical representation of the equation $3x - 4y = 1$ is a straight line. The ordered pairs of real numbers representing the coordinates of any point on the line AB will satisfy this equation. The solution set

$$\{(x, y) / 3x - 4y = 1\}$$

is therefore the set of all points on the straight line $3x - 4y = 1$ and is obviously an infinite set. Thus, any general linear equation of the form $ax + by + c = 0$ has an infinite number of solutions and the solution set is given by

$$\{(x, y)/ax + by + c = 0\}$$

Let us now consider a pair of linear equations $a_1x + b_1y + c_1 = 0$ ...(i)

and $a_2x + b_2y + c_2 = 0$ ...(ii)

The solution sets of these equations are

$$A = \{(x, y)/a_1x + b_1y + c_1 = 0\}$$

and $B = \{(x, y)/a_2x + b_2y + c_2 = 0\}$ respectively.

A being the set of all points on the line (i) and B being the set of all points on the line (ii). If there is a common element (x, y) between A and B then the point which represents this common element lies on both the straight lines (i) and (ii) and is therefore the point of intersection of these two lines.

So, a solution of two simultaneous linear equations is the ordered pair (x, y) which is a common solution of both the equations and hence is a common element of the solution sets A and B of the two equations. Such a solution is therefore given by

$$A \cap B = \{(x, y)/a_1x + b_1y + c_1 = 0 \text{ and } a_2x + b_2y + c_2 = 0\}.$$

Geometrically, since each linear equation represents a straight line and since each solution of the linear equation represents a point on this straight line, therefore the solution common to two linear equations is represented geometrically by the point of intersection of the two straight lines.

Since two independent straight lines intersect at only one point there exists therefore only one solution of two simultaneous linear equations.

5·19. To fix our ideas, let us consider two simultaneous linear equations

$$x + y - 7 = 0$$
$$\text{and } 3x - 4y + 7 = 0$$

The graphs of these equations, shown in the figure, are two lines intersecting at the point (3, 4). The ordered pair (3, 4) therefore satisfies both the equations

$$x + y - 7 = 0$$
$$\text{and } 3x - 4 + 7 = 0$$

simultaneously. The solution of the two given linear equations is therefore, $x = 3$, $y = 4$. In set theoretic notations, if we let

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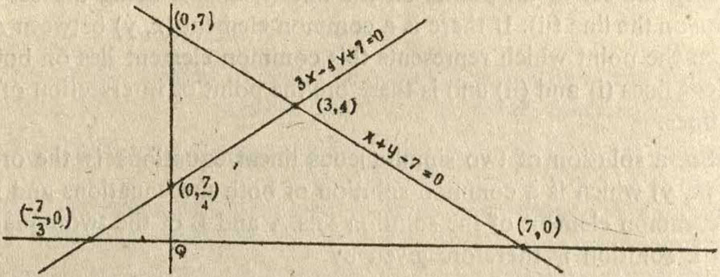
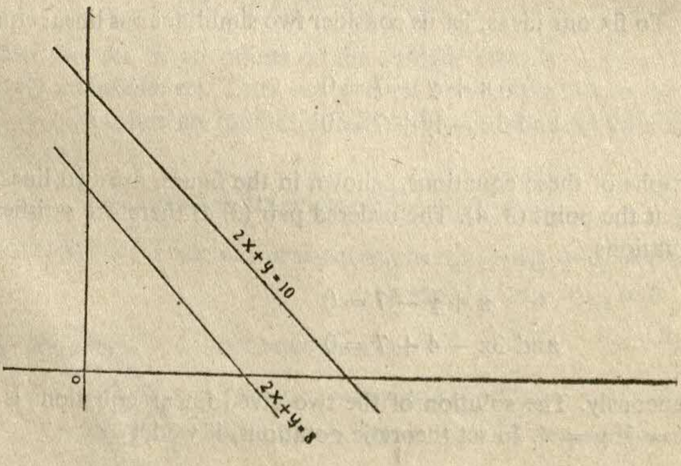


$$A = \{(x, y)/x + y - 7 = 0\}$$
$$\text{and } B = \{(x, y)/3x-4y+7 = 0\}$$

then $A \cap B = \{(3, 4)\}$ and this is the solution set. Note that since two independent straight lines intersect at only one point, the solution set of two consistent, simultaneous linear equations will always be a singleton set.

If the two given linear equations are like $2x + y = 10$
and $2x + y = 8$

then their graphs represent two parallel straight lines which do not interesect. They will not therefore have any common solution. In other

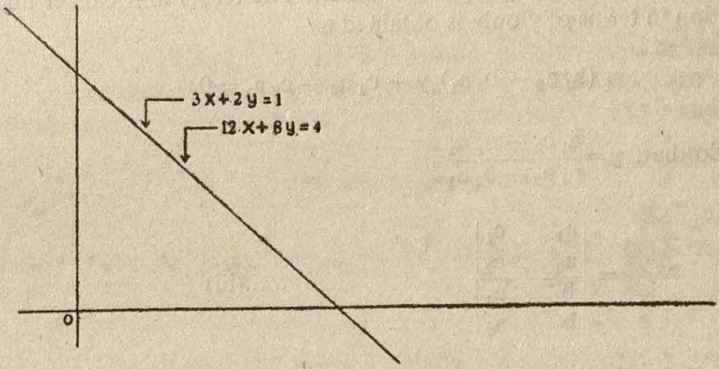


words, the solution set of two such equations is the null set ϕ. As we have already seen while discussing "determinants", two such equations are called inconsistent. Thus the solution set of two inconsistent equations is empty.

If the two given linear equations are of the form

$$3x + 2y = 1$$
$$\text{and } 12x + 8y = 4$$

that is, one of them can be obtained by multiplying the other throughout by a fixed real number, then each of them represents the same straight line. Two such equations are said to be "dependent". Since their graphs



are two coincident lines, each point of one lies on the other and hence they have an infinite number of common points and therefore an infinite number of solutions.

That two dependent linear equations have an infinite number of solutions is to be expected because on dividing one of them throughout by a non-zero real number the other equation is obtained. The two dependent equations are therefore actually the same linear equation written in two different ways and they have an infinite number of solutions just as a single linear equation has.

5·20. The graphical method of finding the solution of two simultaneous linear equations is an approximate method. We proceed now to discuss two other methods of solving two linear equations. One method is through determinants, the other through matrix algebra.

(1) **Determinants :** An accurate method of solving two simultaneous linear equations is that of elimination, a direct formula for which can be devised in terms of determinants of the coefficients of the variables occurring in the two equations.

Consider the two consistent equations:

$$a_1 x + b_1 y + c_1 = 0 \quad \text{......(i)}$$
$$\text{and } a_2 x + b_2 y + c_2 = 0 \quad \text{......(ii)}$$

multiply (i) by a_2, (ii) by a_1 and subtract. The terms in x cancel and an equation in terms of y only is obtained as

$$(b_1 a_2 - b_2 a_1)\, y + c_1 a_2 - c_2 a_1 = 0$$

$$\text{So that, } y = \frac{a_1 c_2 - a_2 c_1}{b_1 a_2 - b_2 a_1}$$

$$= \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}} \quad \text{......(iii)}$$

either by substituting this value of y in one of the given equations or by multiplying (i) by b_2, (ii) by b_1 and subtracting, an equation in terms of x only is obtained as

$$(a_1 b_2 - a_2 b_1)\, x + c_1 b_2 - c_2 b_1 = 0$$

$$\text{that is, } x = \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}$$

$$= \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{......(iv)}$$

From (iii) and (iv) it is evident that

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{-\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

An easy rule for solving the two equations

$$a_1 x + b_1 y + c_1 = 0$$
$$\text{and } a_2 x + b_2 y + c_2 = 0$$

can therefore be formulated as follows :—

$$\frac{x}{\text{The determinant formed by the coefficients other than those of x}} = \frac{y}{\text{The determinant with a change of sign formed by the coefficients other than those of y.}} = \frac{1}{\text{The determinant formed by the coefficients of x and y}}$$

Note that since the given equations are consistent, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ that is $a_1 b_2 - a_2 b_1 \neq 0$.

The method of solving simultaneous linear equations by determinants can be extended to linear equations containing more than two variables. The general method, which is outside the scope of the present discussion, is known as Cramer's rule, named after Gabriel Cramer (1704—1752), a Swiss mathematician.

Illustration :

Solve the equations

$$3x + 4y = 5$$
$$6x + 7y = 12$$

The given equations can be written as

$$3x + 4y - 5 = 0$$
$$6x + 7y - 12 = 0$$

By the determinant rule,

$$\frac{x}{\begin{vmatrix} 4 & -5 \\ 7 & -12 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 3 & -5 \\ 6 & -12 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix}}$$

$$\therefore \quad \frac{x}{-48+35} = \frac{y}{-(-36+30)} = \frac{1}{21-24}$$

$$\therefore \quad \frac{x}{-13} = \frac{y}{6} = \frac{1}{-3}$$

$$\therefore \quad x = \frac{13}{3}, \; y = -2 .$$

(2) **Matrices :**

The equations

$$3x + y = 5$$
$$8x + 5y = 18$$

an be written as

$$\begin{bmatrix} 3 & 1 \\ 8 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \end{bmatrix}$$

premultiply each side by $\begin{bmatrix} 5 & -1 \\ -8 & 3 \end{bmatrix}$ [obtained by the rule : "interchange the elements on the leading diagonal and change the sign of the other two elements without changing their position.]

$$\therefore \begin{bmatrix} 5 & -1 \\ -8 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 8 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -8 & 3 \end{bmatrix} \begin{bmatrix} \\ 18 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

$$\therefore 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 7\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\therefore$ $x = 1$, and $y = 2$.

In general, given any two consistent linear equations

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

they can be represented as

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

On premultiplying each side by $\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}$,

[rule : This matrix is formed from the coefficient matrix, by interchanging the elements in the leading diagonal and changing the signs of the elements in the other diagonal.]

$$\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

So that $(a_1 b_2 - a_2 b_1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_2 c_1 - b_1 c_2 \\ - c_1 a_2 + a_1 c_2 \end{bmatrix}$

$$\therefore (a_1 b_2 - a_2 b_1)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_2 c_1 - b_1 c_2 \\ - c_1 a_2 + a_1 c_2 \end{bmatrix}$$

$$\therefore x = \frac{b_2 c_1 - b_1 c_2}{a_1 b_2 - a_2 b_1} \text{ and } y = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$$

Note that the factor with which both sides are to be multiplied is motivated to reduce the matrix on the left hand side (made up of the coefficients) to the unit matrix. To this end, if the left hand matrix is directly multiplied by its inverse, the same result would be obtained, for, the inverse of $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is $\dfrac{1}{a_1 b_2 - a_2 b_1} \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}$

That is , $\dfrac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}$ and premultiplication by it will reduce the coefficient matrix on the left hand side to the unit matrix, so that the left hand side is reduced to the column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$.

and the right side to the column matrix

$$\frac{1}{a_1 b_2 - a_2 b_1} \begin{bmatrix} b_2 c_1 - b_1 c_2 \\ -c_1 a_2 + a_1 c_2 \end{bmatrix}$$

the result thus obtained being the same as above.

EXERCISE

1. Sketch the graphs of $3x + y + 3 = 0$, $6x + y + 9 = 0$ and find approximately solution of the two linear equations.

2. If $A = \{ (x, y) / 3x - 4y = 1 \}$ and
$B = \{ (x, y) / 6x - 8y = 3 \}$; Draw a graph and find $A \cap B$.

3. Solve the following equations (i) by the method of determinants, (ii) by matrix method.

(i) $3x + 2y = 13$
$5x - y = 6$

(ii) $2x - 3y = 4$
$x + y = 6$

(iii) $3x + 4y = 18$
$2x + 3y = 13$

(iv) $2x - 3y = 1$
$5y - x = 3$

(v) $x = y$
$x + y - 2 = 0$

(vi) $x - y - 4 = 0$
$x + y = 0$

4. Write the following equations in matrix form:

(i) $12x + 5y = 4$
$3x + 4y = 7$

(ii) $2x + y - 4 = 0$
$y + 3x - 5 = 0$

(iii) $7 + 3y - 2x = 0$
$8x - 11y - 33 = 0$

(iv) $2x - 3y + 5 = 0$
$3x + 4y = 0$

(v) $x = 4$
$2x + y = 0$

(vi) $y + 3 = 0$
$5 - x - y = 0$

5. Write each of the following matrix equations as a pair of simultaneous linear equations.

(i) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 0 \\ 2 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

(iv) $\begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 21 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} \\ y \end{bmatrix} = \begin{bmatrix} -18 \\ 16 \end{bmatrix}$

6. Find the coordinates of the vertices of the triangle formed by the lines

   (i) $3x + 5y = 15$, $x + y = 4$, $2x + y = 6$

   (ii) $x - y + 3 = 0$, $3x + 4y = 5$, $6x + y - 17 = 0$

7. What is the value of k if the equations $3x - 5y = 7$ an $9x - 15y = k$ have an infinite number of solutions?

8. If the equations $ax + 36y = 3$ and $4x + 72y = 6$ have an infinite number of solutions, find a.

9. Show that the lines

$$3x - 2y = 5, \quad 2x + y = 7$$
$$3x - 2y = 4, \quad 2x + y = 1$$

form a parallelogram and find its vertices.

ANSWERS

3. (i) $x = \frac{25}{13}$, $y = \frac{47}{13}$.

   (ii) $x = \frac{22}{5}$, $y = \frac{8}{5}$

   (iii) $x = 2$, $y = 3$, (iv) $x = 2$, $y = 1$

   (v) $x = 1$, $y = 1$, (vi) $x = 2$, $y = -2$

4. (i) $\begin{bmatrix} 12 & 5 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

   (iii) $\begin{bmatrix} 2 & -3 \\ 8 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 33 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$

   (v) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$

5. (i) $2x + y = 5$
   $x + 2y = 7$

   (ii) $x - y = 0$
   $3x - y = 4$

(iii) $x = 5$ (iv) $4x = 12$

$2x - 7y = 3$ $-3y = 21$

(v) $-6y = -18$

$x + 5y = 16$

6. (i) $\left(\frac{5}{2}, \frac{3}{2}\right), (2, 2), \left(\frac{15}{7}, \frac{12}{7}\right)$

(ii) $(-1, 2), \ (3, -1), \ (2, 5)$

7. 21

8. 2

9. $(1, -1), \left(\frac{19}{7}, \frac{11}{7}\right), \left(\frac{18}{7}, \frac{13}{7}\right), \left(\frac{6}{7}, -\frac{5}{7}\right)$

5·21. Quadratic Functions :

In art 5·12 of this chapter, we discussed the linear function—an algebraic function in which the defining equation is of the first degree. We now proceed to discuss a second algebraic function the defining equation of which is of the second degree.

Definition : The function $f : x \rightarrow ax^2 + bx + c, \ x \in R$ where a, b, c are real constants and $a \neq 0$, is called a quadratic function.

Before discussing the graph of a general quadratic function $f(x) = ax^2 + bx + c$, we shall consider a few examples.

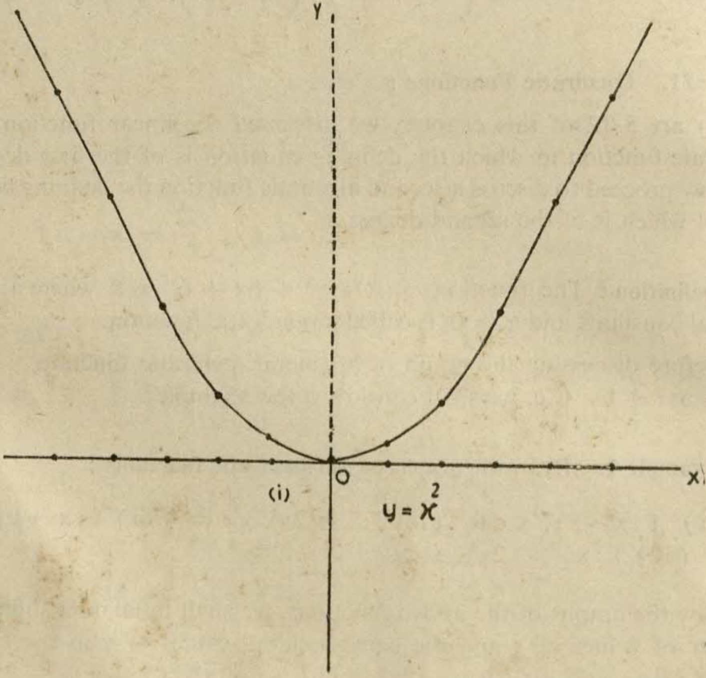
Example 1. Draw the graphs of the quadratic functions :

(i) $f : x \rightarrow x^2, \ x \in R$ (ii) $f : x \rightarrow 2x^2, \ x \in R$ (iii) $f : x \rightarrow \frac{1}{2}x^2$, $x \in R$ (iv) $f : x \rightarrow -2x^2, \ x \in R$

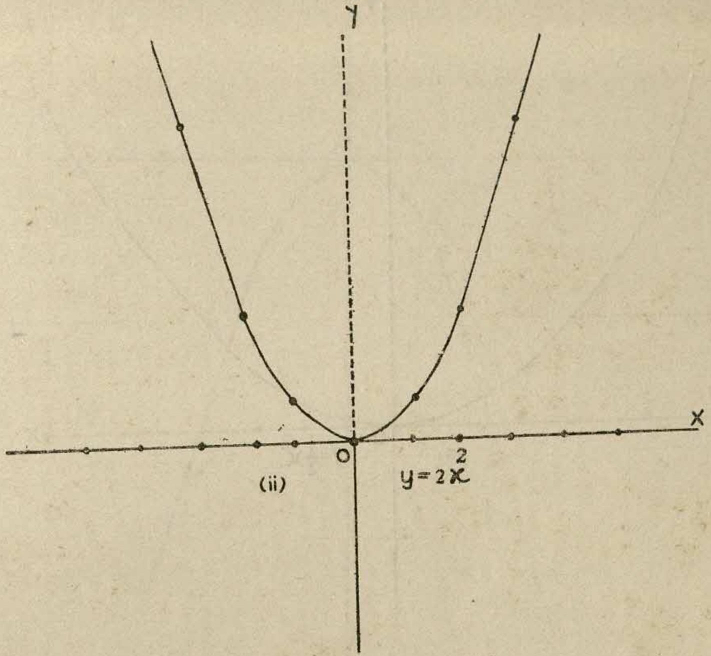
To draw the graphs of the above functions, we shall tabulate a sufficient number of values of x and the corresponding values of y in the equations (i) $y = x^2$, (ii) $y = 2x^2$ (iii) $y = \frac{1}{2}x^2$ (iv) $y = -2x^2$

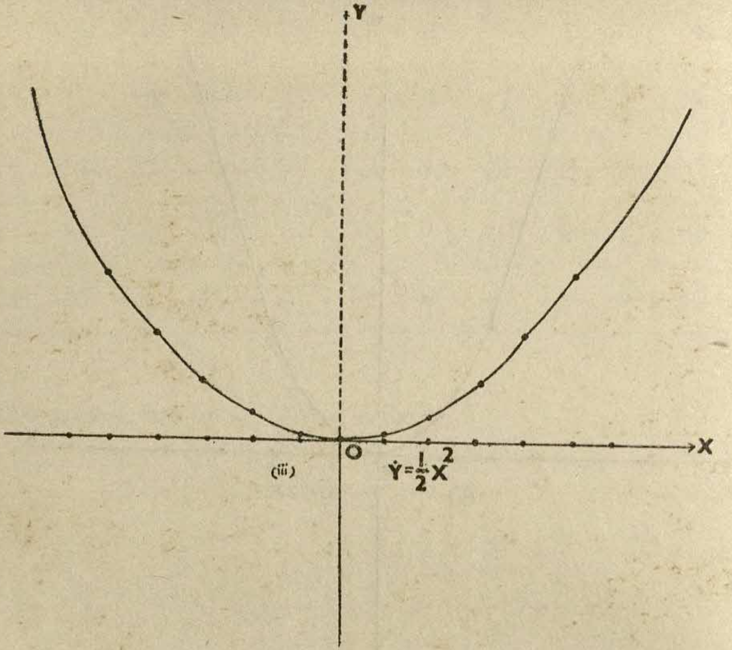
x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$y=x^2$	25	16	9	4	1	0	1	4	9	16	25
$y=2x^2$	50	32	18	8	2	0	2	8	18	32	50
$y=\frac{1}{2}x^2$	$\frac{25}{2}$	8	$\frac{9}{2}$	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	$\frac{9}{2}$	8	$\frac{25}{2}$
$y=-2x^2$	-50	-32	-18	-8	-2	0	-2	-8	-18	-32	-50

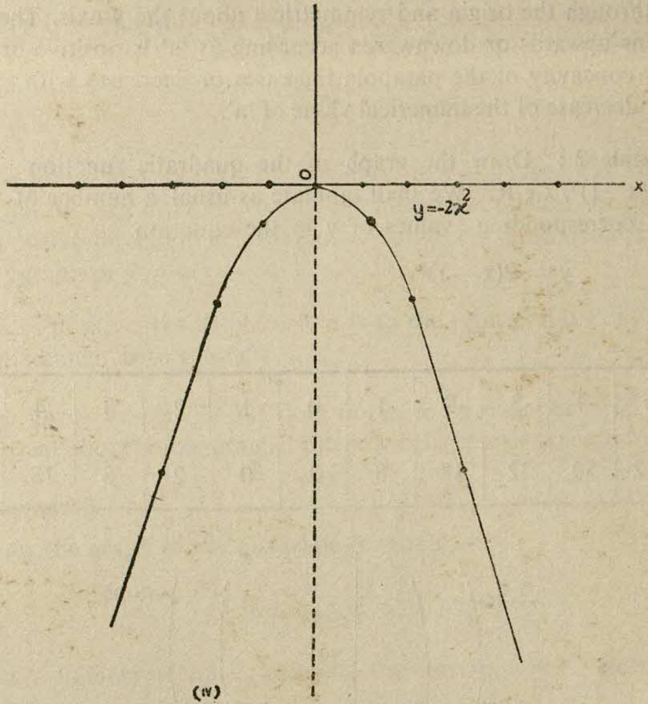
The graphs will be as shown below:



(i) $y = x^2$


Y
X
O
2
(ii)
$y = 2x^2$


Y
X
O
(iii)
$Y = \frac{1}{2} X^2$

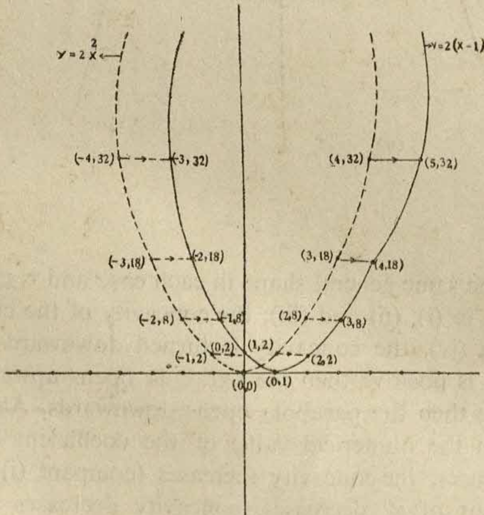


The curve has the same general shape in each case and is called a 'parabola'. Note that in (i), (ii) and (iii), the concavity of the curve is turned upwards and in (iv), the concavity is turned downwards. Thus if the coefficient of x^2 is positive then the parabola opens upwards and if the same is negative then the parabola opens downwards. Also, the concavity depends on the numerical value of the coefficient of x^2. As this coefficient increases, the concavity increases (compare (i) and (ii)) and as the coefficient of x^2 decreases, concavity decreases (compare (iii) with (i) and also with (ii).) Note further that the curve (iv) can be obtained by revolving curve (ii) through two right angles, about the x axis. In general, the graph of $y = ax^2$ (where $a \in R$ is a constant) is a prarabola passing through the origin and symmetrical about the y axis. The parabola opens upwards or downwards according as 'a' is positive or negative. The concavity of the parabola increases or decreases with the increase or decrease of the numerical value of 'a'.

Example 2 : Draw the graph of the quadratic function $f : x \rightarrow 2(x-1)^2,\ x \in R$. We shall tabulate as usual a number of values of x and corresponding values of y in the equation

$$y = 2(x-1)^2$$

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	72	50	32	18	8	2	0	2	8	18	32



The curve is a parabola opening upwards. Comparing with the graph of $y = 2x^2$, we find that the graph of $y = 2(x-1)^2$ is nothing but the graph of $y = 2x^2$ displaced laterally in the positive direction by an amount 1 unit.

Thus the graph of $y = a(x-k)^2$ is similar in every respect to the graph of $y = ax^2$. In fact, if the curve $y = ax^2$ undergoes a lateral displacement in the positive direction by an amount k, the curve obtained coincides with the graph of $y = a(x - k)^2$.

If k is positive, the displacement is to the right and if k is negative the displacement is to the left.

The curve $y = a(x-k)^2$ does not pass through the origin and is symmetrical about a line parallel to the y axis at a distance k from it.

Example 3.

Draw the graph of the quadratic function

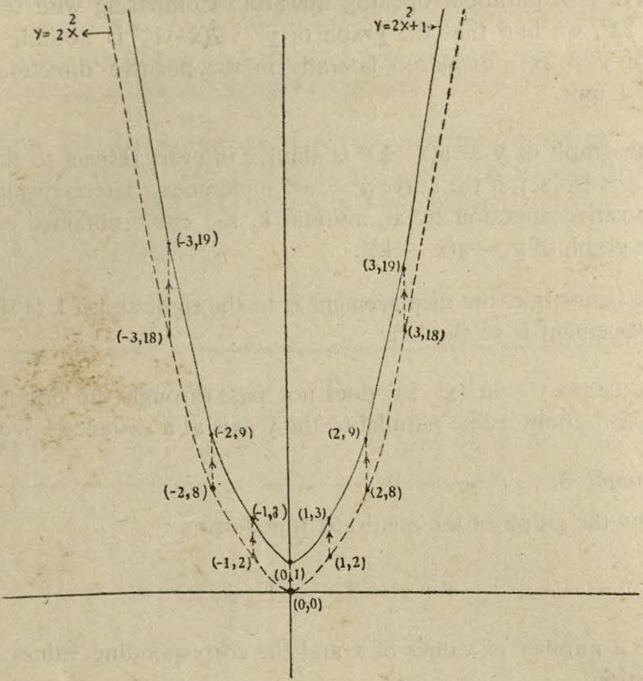
$$f : x \rightarrow 2x^2 + 1, \ x \in R$$

Tabulate a number of values of x and the corresponding values of y in the equation

$$y = 2x^2 + 1$$

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
y	51	33	19	9	3	1	3	9	19	33	51

The graph will be as shown below:



The graph is a parabola again, opening upwards. Comparing with the graph of $y = 2x^2$, we find that the curve $y = 2x^2 + 1$ is the same as the curve that would be obtained if each point of the curve $y = 2x^2$ is displaced vertically in the positive sense by an amount 1 unit.

Thus the graph of $y = ax^2 + b$ is similar to the graph of $y = ax^2$ in every respect and can be obtained by a vertical displacement of the later curve by a distance 'b'.

The curve $y = ax^2 + b$ is symmetrical about y axis but does not pass through the origin.

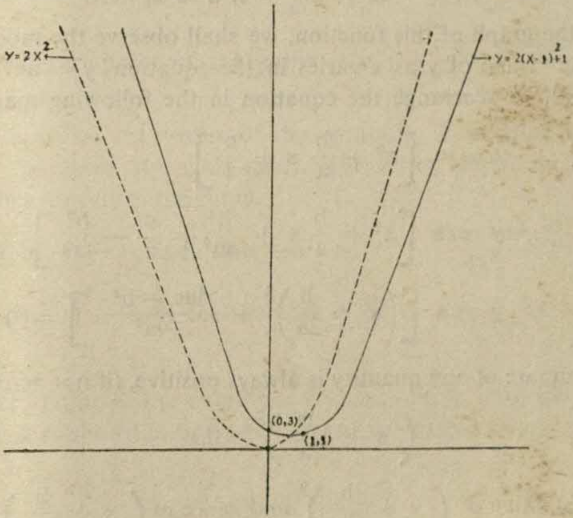
Example 4.

Draw the graph of the quadratic function $f: x \to 2x^2 - 4x + 3$.

We tabulate a sufficient number of values of x and the corresponding values of y in the equation $y = 2x^2 - 4x + 3$. This equation can be written as :

$$y = 2(x - 1)^2 + 1$$

X	-5	-4	-3	-2	-1	0	1	2	3	4	5
Y	73	51	33	19	9	3	1	3	9	19	33



The graph obtained is a parabola opening upward. Compare with the graph of $y = 2x^2$. You will find that if the curve $y = 2x^2$ is displaced laterally by an amount 1 unit and vertically by an amount 1 unit, then t will coincide with the graph of $y = 2(x-1)^2 + 1$.

In general, every quadratic function

$$f : x \rightarrow ax^2 + bx + c, \quad x \in R$$

that is, $y = ax^2 + bx + c$ can be written as

$$y = a\,(x - k)^2 + d$$

where k and d are constants depending upon the values of a, b and c.

The graph of such a function is a parabola and is the same as the graph of $y = ax^2$ with a lateral shift k and vertical shift d.

The curve is always symmetric about a line parallel to the y axis and situated at a distance k from the y axis. The vertex is situated at the point (k, d.).

5·22. Let us now consider the general quadratic function

$$f : x \to ax^2 + bx + c,\ a \neq 0,\ x \in R.$$

To draw the graph of this function, we shall observe the mode of variation of the value of y as x varies in the equation $y = ax^2 + bx + c$. To this end, we rearrange the equation in the following manner :

$$y = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

$$\therefore\ y = a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

$$\therefore\ y = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] \quad \ldots(i)$$

Since the square of any quantity is always positive, (if not zero) therefore

$$\left(x + \frac{b}{2a}\right)^2 \geqslant 0.$$

So the least value of $\left(x + \frac{b}{2a}\right)^2$ and hence of $\left(x + \frac{b}{2a}\right)$ is zero.

$\therefore$ The least value of the quantity in the square bracket in right hand side of (1) occurs for that value of x, for which

$$x + \frac{b}{2a} = 0 \text{ i.e. at } x = -\frac{b}{2a}$$

Whether the whole expression in r.h.s. of (1) (and hence the value of y) does or does not become minimum at this value of x depends on whether a is +ve or −ve. We consider it therefore under the following cases.

Case 1 : $a > 0$

At $x = -\frac{b}{2a}$, the quantity in the bracket in r.h.s. of (i) is minimum, and since $a > 0$ the value of the whole expression in R. H. S. of (1) is minimum.

Thus, at $x = -\frac{b}{2a}$, y is minimum.

But at $x = -\frac{b}{2a}$, $y = \frac{4ac - b^2}{4a}$

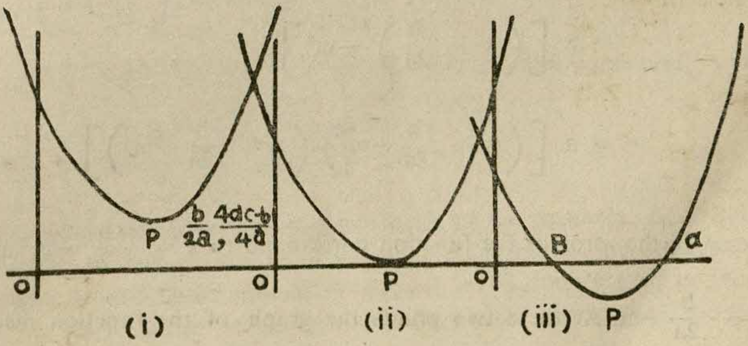
$\therefore$ The lowest point of the graph is at $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$.

This point is called the vertex of the parabola. Remember that when $a > 0$, the parabola opens upwards. The vertex determined, we turn to some other aspects of the curve.

(i) If $b^2 < 4ac$, then $4ac - b^2 > 0$ $\therefore$ $\frac{4ac - b^2}{4a^2} > 0$; also, since $\left(x + \frac{b}{2a}\right)^2 \geqslant 0$, therefore at $a > 0$, $ax^2 + bx + c > 0$

$\therefore y > 0$.

That is, every point of the curve has a y co-ordinate which is positive. $\therefore$ the curve is as shown in fig (i) below, and is situated completely above the axis of x.



(ii) If $b^2 = 4ac$, then $4ac - b^2 = 0$

$$\therefore y = a\left(x + \frac{b}{2a}\right)^2$$

$$\therefore y \geqslant 0 \left[\because a > 0 \text{ and } \left(x + \frac{b}{2a}\right)^2 \geqslant 0\right]$$

The zero of the function is at $y = 0$ which corresponds to $x = -\dfrac{b}{2a}$ i.e. at the vertex of the parabola. For all other points, $y > 0$; so all other points are situated above the x-axis. The graph of the function is as shown in figure (ii) above and is a parabola opening upwards, situated completely above x axis except at the single point $\left(-\dfrac{b}{2a}, 0\right)$ where it meets the x axis. Actually, the parabola touches the x axis a this point.

(iii) If $b^2 > 4ac$, then $b^2 - 4ac > 0 \quad \therefore \dfrac{b^2 - 4ac}{4a^2} > 0.$

$$\text{Let } \frac{b^2 - 4ac}{4a^2} = m^2$$

$$\therefore y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - m^2\right]$$

$$= a\left[\left(x + \frac{b}{2a} + m\right)\left(x + \frac{b}{2a} - m\right)\right]$$

$\because a \neq 0$, the zeros of the function correspond to $x = -\dfrac{b}{2a} - m$ and $x = -\dfrac{b}{2a} + m$. At these two points the graph of the function meets the axis of x.

Denote $\left(-\frac{b}{2a}-m\right)$ by β and $\left(-\frac{b}{2a}+m\right)$ by $\propto$.

Then, $y = a\,(x-\propto)\,(x-\beta)$; $(\beta<\propto)$.

The graph of the function is a parabola which opens upwards, meets the x axis at β and $\propto$ and has the vertex at $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$. Note that y is +ve at all points except between $\propto$ and β. For, when $\beta < x < \propto$, $(x-\propto)$ is $-$ ve, $(x-\beta)$ is + ve. So, since $a > 0$, $a\,(x-\propto)\,(x-\beta)$ is $-$ve $\therefore$ y is $-$ve. So the portion of the graph between $\propto$ and β is situated below the x$-$axis. For all other points, the graph lies above the axis of x.

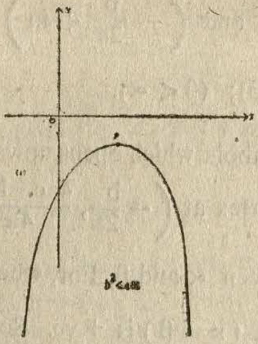
In all cases the curve is symmetrical about $x = -\frac{b}{2a}$ and the lowest point is $A\left\{-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right\}$

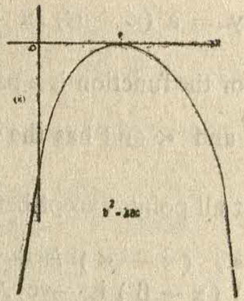
Case II : $a < 0$

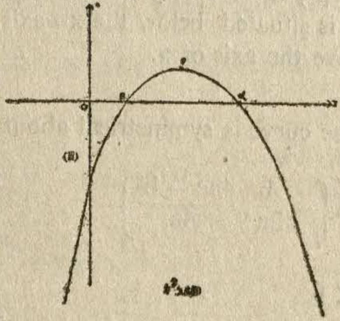
The parabola opens downwards. At $x = \frac{-b}{2a}$, the minimum value of $\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}$ occurs. Since $a<0$ this corresponds to the maximum value of

$$a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right],$$

i .e. of y. This point is called the vertex of the parabola. As in case I, in this case also, three possibilities arise. It is left for the students to elaborate upon these subcases which is to be done exactly as in case I The only change necessary when $a<0$ is a change in the sign of y. The graphs are as shown below.







Note that when $a<0$ and $b^2>4ac$, the function ax^2+bx+c is +ve only when $\beta < x < \alpha$; so that the graph of the function between β and α is situated above the x-axis. At all other points, the graph lies below the axis of x.

5·23. The quadratic equation:

We shall next elaborate upon the zeros of a quadratic function. We know that the zeros of a function $y = f(x)$ are those values of x for which y is equal to zero. So, the zeros of the quadratic function $y = ax^2 + bx + c$ are those values of x for which $y = 0$; that is, $ax^2 + bx + c = 0$.........(i)

We shall solve this equation to obtain the values of x satisfying it. The equation $ax^2 + bx + c = 0$ is the general quadratic equation, and may be solved as under :

$$ax^2 + bx + c = 0$$

$$\therefore\ x^2 + \frac{b}{a}x = -\frac{c}{a}\ \left(\because a \neq 0\right)$$

$$\therefore\ x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\therefore\ \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\therefore\ x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, there are two zeros of the quadratic function. These are also the values of x that satisfy the general quadratic equation, and are called the roots of the quadratic equation. In short, the zeros of the quadratic function ax^2+bx+c are the roots of the quadratic equation $ax^2+bx+c=0$, and if these roots be denoted by α and β, then

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and}$$

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

In the discussion made hereafter, α and β will always denote the two roots as given above, of the quadratic equation $ax^2 + bx + c = 0$, unless otherwise specifically stated.

5·24. Nature of the roots of the equation $ax^2 + bx + c = 0$.

In the equation $ax^2 + bx + c = 0$, $(a \neq 0)$, a, b, c are any real constants. The values and hence the nature of the roots α and β depend exclusively upon the relative values of a, b and c. The following are three major possibilities that arise.

(i) $b^2 > 4ac$

$b^2 - 4ac$ is positive $\therefore \sqrt{b^2 - 4ac}$ exists and is real. Therefore α and β exist and are unequal.

Compare with article 5·22, figure (iii) of case I and case II. Note that in both cases, $b^2 > 4ac$ corresponds to a graph which intersects the axis of x at two real and different points α and β.

(ii) $b^2 = 4ac$

$b^2 - 4ac = 0$

$$\therefore \alpha = -\frac{b}{2a} \text{ and } \beta = -\frac{b}{2a}$$

Thus α and β exist and are equal. Compare with article 5·22, fig. (ii) of both the cases I and II. In both cases, $b^2 = 4ac$ gives rise to a graph which meets the x axis at a single point (that is, two coincident points).

(iii) $b^2 < 4ac$

$\therefore b^2 - 4ac$ is negative.

$\therefore \sqrt{b^2 - 4ac}$ cannot be found in the system of real numbers.

$\therefore$ Neither α nor β exists, and hence the roots do not exist in the real number system. Compare again with art. 5·22 fig. (i) of case I and II. In both cases, $b^2 < 4ac$ corresponds to a graph which does not meet the axis of x. In other words, the zeros of the functions under this condition do not exist. There is a form of algebra called complex algebra in which the expression for α and β, when they do not exist in the system of real numbers, have an interpretation. α and β are then said to be "imaginary". We shall not, however, go into it in the scope of our discussions.

5·25. Relations between the roots and coefficients of the quadratic equation.

The roots of the general quadratic equation $ax^2+bx+c=0$ are given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Therefore,

adding, $\alpha + \beta = -\frac{b}{a}$ .... (i)

and multiplying, $\alpha\beta = \frac{c}{a}$ ....(ii)

The equation $ax^2 + bx + c = 0$ can be expressed in terms of its roots α and β as :

$$ax^2 + bx + c = 0$$

$$\therefore\ a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = 0$$

$$\because a \neq 0,\ \therefore\ x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\therefore\ x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a} = 0$$

$$\therefore\ x^2 - (\alpha + \beta)x + \alpha\beta = 0 \quad .....(iii)$$

$$\therefore\ (x - \alpha)(x - \beta) = 0 \quad ....(iv)$$

The equation (iii) shows that a qudratic equation can be written as x^2—(sum of the roots) x + (product of the roots) = 0

The results (i), (ii), (iii), (iv) are very useful in finding the roots or constructing a quadratic equation and are to be remembered.

Illustration :

The roots of the equation $x^2 - 3x + 5 = 0$ are α and β. Find the equation whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$

Since α and β are the roots of $x^2 - 3x + 5 = 0$

$\therefore$ from (i) and (ii) above,

$$\alpha + \beta = 3$$
$$\alpha\beta = 5.$$

We wish to form a quadratic equation whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$. For this equation, sum of the roots $= \alpha + \frac{1}{\beta} + \beta + \frac{1}{\alpha}$.

$$= (\alpha + \beta) + \left(\frac{1}{\beta} + \frac{1}{\alpha}\right)$$
$$= (\alpha + \beta) + \frac{\alpha + \beta}{\alpha\beta}$$
$$= 3 + \frac{3}{5} = \frac{18}{5}.$$

Product of the roots $= \left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right)$

$$= \alpha\beta + 2 + \frac{1}{\alpha\beta}$$
$$= 5 + 2 + \frac{1}{5}$$
$$= \frac{36}{5}$$

∴ From the formula (iii) of this article, the required equation is

$$x^2 - \frac{18}{5}x + \frac{36}{5} = 0$$

i.e. $5x^2 - 18x + 36 = 0.$

5·26. A quadratic equation cannot have more than two roots :

If possible, let the quadratic equation $ax^2+bx+c=0$ have three distinct roots α, β and γ.

$$\therefore \quad a\alpha^2 + b\alpha + c = 0 \qquad \text{...(i)}$$
$$a\beta^2 + b\beta + c = 0 \qquad \text{...(ii)}$$
$$a\gamma^2 + b\gamma + c = 0 \qquad \text{...(iii)}$$

from (i) and (ii), subtracting,

$$a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0$$

$$\therefore\ (\alpha - \beta)[a(\alpha + \beta) + b] = 0$$

Since α and β are distinct roots, $\alpha \neq \beta$.

$$\therefore\ a(\alpha + \beta) + b = 0 \quad \ldots\text{(iv)}$$

Similarly from (ii) and (iii),

$$a(\beta + \gamma) + b = 0 \quad \ldots\text{(v)}$$

From (iv) and (v), Subtracting,

$$a(\alpha - \gamma) = 0$$

Since $a \neq 0$, $\therefore\ \alpha - \gamma = 0 \quad \therefore\ \alpha = \gamma$.

This contradicts the assertion that α and γ are distinct roots.

So, a quadratic equation cannot have more than two distinct roots.

5·27. If one root of a quadratic equation with rational coefficients be irrational, the other root must be its irrational conjugate. Let one root of the quadratic equation $ax^2 + bx + c = 0$ (where a, b, c are rational and $a \neq 0$) be $m + \sqrt{n}$ where m is rational and $\sqrt{n}$ is irrational. Let the other root of the equation be β. We wish to show that $\beta = m - \sqrt{n}$.

$$\text{Now, } \beta + (m + \sqrt{n}) = -\frac{b}{a} \quad \ldots\ \text{(i)}$$

$$\text{and } \beta(m + \sqrt{n}) = \frac{c}{a} \quad \ldots\ \text{(ii)}$$

$$\text{from (i), } \beta = \frac{-b}{a} - m - \sqrt{n}$$

$$= k - \sqrt{n} \text{ where } k = -\frac{b}{a} - m$$

$$\text{from (ii), } (k - \sqrt{n})\left(m + \sqrt{n}\right) = \frac{c}{a}$$

$$\therefore \; km - n + (k - m)\sqrt{n} = \frac{c}{a}.$$

The right hand side of this equation is purely rational; hence we must have

$$(k - m)\sqrt{n} = 0$$

$$\therefore \; k = m$$

Hence $\beta = m - \sqrt{n}$

EXERCISE

1. Find the roots of the following quadratic equations :

(i) $6x^2 - 5x - 50 = 0$, (ii) $x^2 + 2ax + a^2 - b^2 = 0$
(iii) $4x^2 - x = 0$, (iv) $x^2 - 3 = 4x + 1$
(v) $3x + 2 + 4(x^2 - 1) = 0$ (vi) $3(x^2 + 1) = x + 5$
(vii) $2x^2 + x - 12 = 0$.

2. Form the quadratic equation in each of the following cases, the roots being

(i) $-\frac{1}{2}, \frac{6}{13}$; (ii) $-5, 4$; (iii) $\frac{m}{n}, -\frac{n}{m}$;
(iv) $2 + \sqrt{3},\ 2 - \sqrt{3}$; (v) $7 \pm 2\sqrt{5}$

3. The sum of a number and its reciprocal is $\frac{34}{15}$. Find the number.

4. For what values of m will the equation $x^2 - 2x(1 + 3m) + 7(3 + 2m) = 0$ have equal roots?

5. For what values of m will the equation $9x^2 - (2m^2 + 5m - 12)x = 34$ have roots which are equal in magnitude but opposite in sign?

6. Find the condition that one root of the quadratic equation $ax^2 + bx + c = 0$ will be three times the other.

7. If $(6-x)$, $(13-x)$, are the lengths of the sides of a right angled triangle whose hypotenuse is $(14-x)$, find the value of x.

8. Form a quadratic equation whose roots are the squares of the roots of $2x^2+x-6=0$.

9. If the roots of the equation

$$(p^2+q^2)x^2-2(ap+bq)x+(a^2+b^2)=0 \text{ are equal,}$$

then show that $\frac{a}{b}=\frac{p}{q}$

10. If α and β be the roots of the equation $3x^2+5x-4=0$, find the value of

(i) $\alpha^2+\beta^2$; (ii) $\alpha-\beta$; (iii) $\alpha^2-\beta^2$; (iv) $\alpha^3+\beta^3$;

(v) $\alpha^3-\beta^3$; (vi) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$; (vii) $\frac{1}{\alpha^3}+\frac{1}{\beta^3}$

(viii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

11. If the roots of $x^2-px+q=0$ be α and β, then prove that

$$\frac{1}{\alpha^3}+\frac{1}{\beta^3}=\frac{p^3}{q^3}-\frac{3p}{q^2}$$

12. If α, β are the roots of the equation $2x^2+4x-1=0$ find the equations whose roots are (i) $3\alpha-5$, $3\beta-5$; (*ii*) α^2, β^2,;

(iii) $\frac{1}{\alpha}+5$, $\frac{1}{\beta}+5$.

13. If the ratio of the roots of $ax^2+bx+c=0$ be k, show that

$$\frac{m^2+1}{m}=\frac{b^2-2ac}{ac}$$

14. The difference between the roots of the quadratic equation $3x^2-7x+k=0$ is 2. Find the value of k.

15. If the roots of $ax^2+bx+c=0$ be in the ratio p : q, show

that $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{a}{c}}=0$

16. Show that the roots of the equation $x^2 - (p + 2)x + p = 0$ are real for all real values of p.
17. The roots of the equation $x^2 + lx + m = 0$ differ by 3. Prove that $l^2 = 4m + 9$
18. If one root of the equation $2x^2 + ax - 15 = 0$ equals 3, find the value of a.
19. If one root of the equation $5x^2 - 4x + m = 0$ is equal to zero, find the other root. Find also the value of m.
20. The product of two consecutive positive integers is 30. Find the integers.

ANSWERS

1. (i) $1, -\frac{5}{4}$; (ii) $-a \pm b$; (iii) $0, \frac{1}{4}$; (iv) $2\left(1 \pm \sqrt{2}\right)$;

   (v) $\frac{1}{8}\left(-3 \pm \sqrt{41}\right)$; (vi) $1, -\frac{2}{3}$; (vii) $\frac{1 \pm \sqrt{97}}{4}$;

2. (i) $26x^2 + x - 6 = 0$; (ii) $x^2 + x - 20 = 0$

   (iii) $mnx^2 + (n^2 - m^2)x - mn = 0$; (iv) $x^2 - 4x + 1 = 0$;

   (v) $x^2 - 14x + 29 = 0$.

3. $\frac{3}{5}$ or $\frac{5}{3}$

4. $2, -\frac{10}{9}$

5. $\frac{3}{2}, -4$

6. $3b^2 = 16ac$

7. 1

8. $4x^2 - 25x + 36 = 0$

10. (i) $\frac{49}{9}$; (ii) $\frac{\sqrt{73}}{3}$;(iii) $\frac{49\sqrt{73}}{27}$; (iv) $-\frac{305}{27}$; (v) $\frac{37\sqrt{73}}{27}$;

(vi) $\frac{49}{16}$; (vii) $\frac{305}{64}$; (iii) $-\frac{49}{12}$

12. (i) $2x^2 + 32x + 101 = 0$; (ii) $4x^2 - 20x + 1 = 0$;
(iii) $x^2 - 14x + 43 = 0$.

14. $\frac{13}{12}$

18. -1

19. $\frac{4}{5}, 0$

20. 5, 6

CHAPTER SIX

Exponential Function, Geometric Progression, Logarithmic Function

6·1. In the previous chapter we have discussed some simple algebraic functions. We proceed now to discuss in this chapter two more functions, the exponential and the logarithmic functions and some of their properties and applications. Throughout the discussion, unless otherwisee specifically mentioned, the domain of definition will be R, the set of all real numbers.

6·2. The exponential function :

Definition: The function $f : x \rightarrow a^x$, $x \in R$ and $a > 0$ is called the exponential function.

'a' is called the base of the exponential function and is an arbitrary constant. We shall assume that the laws of indices for irrational index are the same as the laws for rational indices. In other words, whether the power 'x' of 'a' is rational or irrational, the properties of the exponential function will remain unaltered.

The exponential function $f : x \rightarrow a^x$ must not be confused with the power function $f : x \rightarrow x^n$. In the former, the base is an arbitrary constant and the index is variable whereas in the latter the base is variable while the index remains constant. The properties of the power function are quite different from those of the exponential function.

In the exponential function $f : x \rightarrow a^x$ the rule of function can be written as usual as $y = a^x$, the symbol y standing for the image. The domain of definition (that is, the set of values of x) as already stated is the set of all real numbers; 'a' is a positive real number.

The value of a^x and hence the value of image y is real. Thus the co-domain of an exponential function is the set R of all real numbers and the range is a subset of R. Thus any object x and its corresponding image y form an ordered pair (x, y) in which each component is a real number. These ordered pairs can therefore be geometrically represented by points in a two-dimensional cartesian reference frame.

6·3. We shall discuss this function through its graph under three cases :

Case I : $a > 1$

To fix our ideas, suppose $a = 2$.
Then the function is defined by the rule $y = 2^x$, $x \in R$.
To draw the graph of this function we consider the values of a few ordered pairs (x, y) choosing some particular real values for x and obtaining the corresponding values of y from the above rule.

x	-4	-3	-2	-1	0	1	2	3	4
y	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

We can, of course, take as many real values of x as we desire, get the corresponding values of y, and plot the ordered pairs (x, y) in the two-dimensional rectangular reference frame. However, the nature of the function (and hence of the graph) can be determined by observing the mode of variation in y with respect to the mode of variation in x. Studying the pairs of values tabulated above, we make the following observations :

1. Whatever be the value of x, +ve, zero, or −ve, the value of y is always +ve. Therefore the graph must lie completely in the +ve side of the y-axis, that is, in the 1st and 2nd quadrants.

There will not exist any portion of the graph in the 3rd and 4th quadrants, that is, below the x axis.

2. As x increases, y increases faster and as x decreases, y also decreases at a faster rate. The function is therefore an increasing function.

3. As x decreases having —ve values, y decreases and approaches the value zero.

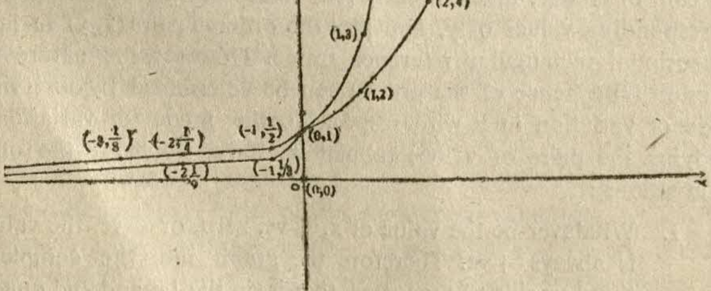
   But y can never actually be equal to zero, for no real value of x can make 2^x equal to zero. Thus the graph of the function will be nearer and nearer to the negative side of the x -axis as x decreases having —ve values, but it will never meet the x axis.

4. As y is never equal to zero, the function has no zeros. The graph will not cross the x-axis at any point.

5. The point (0, 1) lies on the graph.

   These observations will hold whatever be the value of 'a', provided a > 1. The table of values of the ordered pairs will of course be different for different values of a.

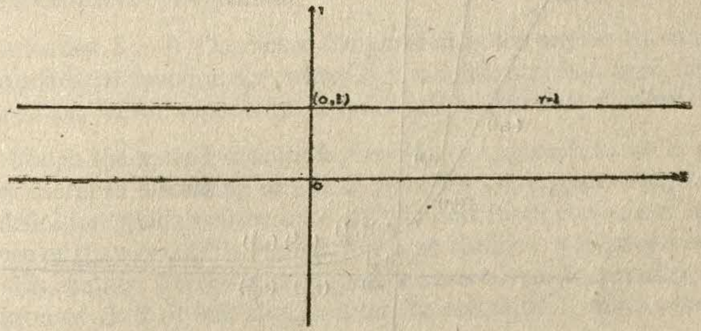
From observation 5 made above, we see that whatever be the



value of a, a^x will always be equal to 1 for $x = 0$. Thus (0, 1) lies on the graph of every exponential function. All the exponential graphs therefore meet at the point (0, 1). In the diagram are shown two exponential graphs corresponding to the values $a = 2$ and $a = 3$.

Case II : $a = 1$

The rule, or the function, becomes $y = 1^x = 1$. Notice that this is a constant function. So, when $a = 1$, the exponential function is reduced to a constant function. Its graph is therefore a straight line passing through (0, 1) and parallel to the x-axis. It is left for the students to make the table.



Verify that all the observations made for case I hold in this case except observation 3. The function now is a constant function, instead of being an increasing function.

Case III : $a < 1$

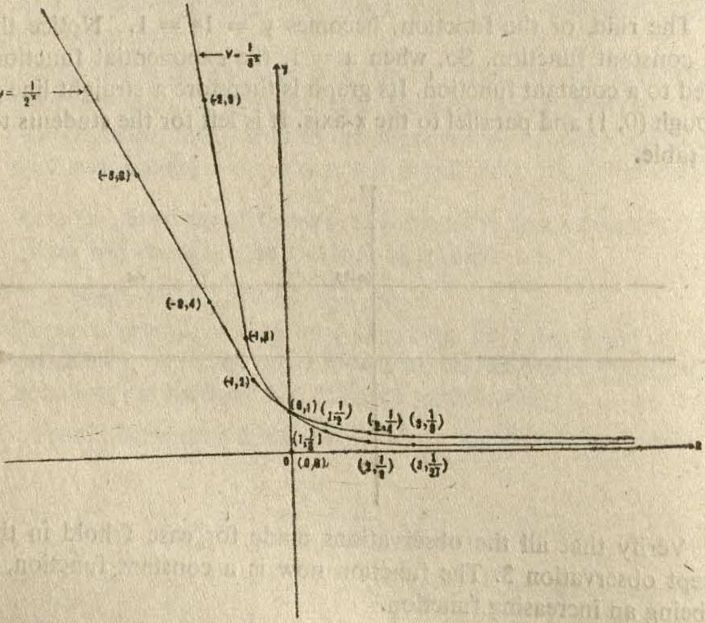
To fix our ideas we assume $a = \frac{1}{2}$ and form the following table

$$y = \frac{1}{2^x}$$

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	32	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$

Verify again that all observations hold as in case I, except observations 2 and 3. In this case, the function is a decreasing function.

As x increases with +ve values, y decreases fast. The graph therefore approaches nearer and nearer to the positive side of the x-axis but never meets it. The graph is as shown in the figure.



In both the cases I and III, x-axis is said to be an 'asymptote' to the graph.

6·4. Geometrical Progression :

Consider the exponential function $f : x \rightarrow 2^x$, $x \in I^+$, where I^+ is the set of all +ve integers. As x assumes the successive values 1, 2, 3 . . . . the images corresponding to these objects are obtained as $2^1, 2^2, 2^3, 2^4, \ldots$. that is 2, 4, 8, 16, ...

This is a series in which every term (except the first) can be obtained by multiplying the previous term by 2.

Similarly, the function $f : x \rightarrow 3^x$, $x \in I^+$ has the sequential range (refer to art 5·13, ch. 5)

$3^1, 3^2, 3^3, 3^4 \ldots \ldots \ldots \ldots$

that is $3, 9, 27, 81 \ldots \ldots \ldots \ldots$

which is a series in which each term (except the first) can be obtained by multiplying the previous term by 3.

Such sequences as mentioned above are called *Geometrical Progression* or G. P. The two following definitions of G. P. are equivalent.

Definition 1. A sequence in which each term after the first is obtained by multiplying the preceding term with a fixed number, is called a Geometric Progression.

Definition 2. A Geometric Progression is the sequential range of any exponential function Kr^u where K is a constant, r is a fixed number and u is any linear expression in terms of the variable x, where $x \in I^+$.

Though the second definition provides an approach to G. P. which is very useful in discussing certain aspects like convergence etc., it is the first definition which will be both simpler and more convenient for the purpose of the scope of discussion which we shall be restricting ourselves to in this chapter. Therefore, whenever reference is made hereafter of the definition of G. P. it will always mean the definition 1 unless otherwise specifically stated.

6·5. From the above definition, it is clear that a geometric progression must have a starting term called the first term, and a fixed number with which a term should be multiplied in order to obtain the next term. This fixed number is called the common ratio.

Thus, in art 6·4, in the first sequence, the first term is 2 and the common ratio is also 2. In the second sequence of the same art., the first term is 3 and the common ratio is 3. In general, we shall use t_1 to denote the first term and r to denote the common ratio of the general G. P. $t_1, t_1r, t_1r^2, t_1r^3 \ldots \ldots \ldots \ldots$

The n^{th} term :

Let t_1 be the first term and r the common ratio of a G. P. which consists of n terms the nth term being denoted by t_n. Then, the formula for t_n can be obtained as follows :

Method I : $\frac{t_2}{t_1} = r$, $\frac{t_3}{t_2} = r$, $\frac{t_4}{t_3} = r$

and so on, $\frac{t_n}{t_{n-1}} = r$.

Multiplying all these equations,

$$\frac{t_2}{t_1} \times \frac{t_3}{t_2} \times \frac{t_4}{t_3} \times \ldots\ldots \times \frac{t_n}{t_{n-1}} = r.\,r.\,r \ldots.\ (n-1) \text{ times}$$

$$\therefore \frac{t_n}{t_1} = r^{n-1} \quad \therefore t_n = t_1 \cdot r^{n-1}.$$

Method II : (the method of induction)

We shall now prove the formula $t_n = t_1 r^{n-1}$ by the method of induction.

Here, P_n is, $t_n = t_1 r^{n-1}$.

$\therefore$ P(1) is, $t_1 = t_1 r^{1-1} = t_1$ which is true.

$\therefore$ P(1) is true ....(i)

Let us assume P(m)

So, $t_m = t_1 r^{m-1}$ ......(ii)

Since the terms are of a G. P. with common ratio r,

$$\therefore \frac{t_{m+1}}{t_m} = r$$

$$\therefore t_{m+1} = t_m \times r$$

$$\therefore \text{from (ii)}, t_{m+1} = t_1 r^{m-1} \times r$$

$$= t_1 r^m$$

But $t_{m+1} = t_1 r^m$ is nothing but P(m+1). Thus P(m) implies P(m+1) ....(iii)

From (i) and (iii), P(1) is true and P(m) implies P(m+1).

Therefore $t_n = t_1 r^{n-1}$ is true for all positive integral values of n.

Example 1. Find the 12^{th} term of the series $-1, 2, -4, 8, -16\ldots$

This is a geometric progression in which $t_1 = -1$, $r = \frac{2}{(-1)} = -2.$

Therefore, the 12^{th} term is given by

$$\begin{aligned} t_{12} &= (-1)\,(-2)^{12-1} \\ &= (-1)\,(-2)^{11} \\ &= 2048. \end{aligned}$$

Example 2. Find the 10^{th} term of a G. P. whose 3^{rd} and 6^{th} terms are 1 and $-\frac{1}{8}$ respectively.

Let t_1 denote the first term and r denote the common ratio of the G. P.

It is given that $t_3 = 1$ and $t_6 = -\frac{1}{8}$. Using the formula for the n^{th} term,

$$1 = t_1\, r^2 \ldots\ldots(i)$$

$$\text{and} \quad -\frac{1}{8} = t_1\, r^5 \ldots\ldots(ii)$$

Dividing (ii) by (i), $r^3 = -\frac{1}{8}$

$$\therefore r = -\frac{1}{2}$$

Substituting in (i),

$$t_1 \left(-\frac{1}{2}\right)^2 = 1$$

$$\therefore t_1 = 4.$$

The 10^{th} term will be given by

$$\begin{aligned} t_{10} &= t_1\, r^{10-1} \\ &= 4\cdot\left(-\frac{1}{2}\right)^9 \\ &= -\frac{1}{128} \end{aligned}$$

6·6 The sum of a G. P.

Let S_n denote the sum of a geometric progression having n terms whose first term is t_1 and common ratio is r.

Then, $S_n = t_1 + t_1 r + t_1 r^2 + \ldots\ldots + t_1 r^{n-2} + t_1 r^{n-1}$

$\therefore\ r.S_n = t_1 r + t_1 r^2 + \ldots\ldots + t_1 r^{n-2} + t_1 r^{n-1} + t_1 r^n$

Subtracting the terms of the second equation from those of the first, we have

$S_n - rS_n = t_1 - t_1 r^n$; all other terms cancelling in pairs.

$$\therefore\ S_n (1-r) = t_1 (1-r^n)$$

$$\therefore\ S_n = \frac{t_1 (1-r^n)}{1-r},\ (r \neq 1) \ldots..(1)$$

Obviously, this formula can also be written as

$$S_n, = \frac{t_1 (r^n-1)}{r-1}\ ;\ (r \neq 1) \ldots.(2)$$

Since $t_n = t_1 r^{n-1}$,

therefore the sum to n terms of the G. P can be written as

$$S_n = \frac{t_1 (1-r^n)}{(1-r)}$$

$$= \frac{t_1 - t_1 r^{n-1} r}{1-r}$$

$$= \frac{t_1 - t_n \cdot r}{1-r},\ (r \neq 1) \ \ldots\ldots\ldots\ldots\ldots(3)$$

What happens when $r = 1$?

Example 1. The sum of the first 6 terms of a G. P. is 9 times the sum of the first 3 terms. Find the common ratio.

Let t_1 be the first term and r be the common ratio.

The sum of the first 6 terms, i. e.

$$S_6 = \frac{t_1 (1 - r^6)}{(1 - r)}$$

and the sum of the first 3 terms i. e.

$$S_3 = \frac{t_2 (1 - r^3)}{(1 - r)}$$

$$\text{Since } S_6 = 9\, S_3$$

$$\therefore \frac{t_1(1 - r^6)}{1 - r} = 9 \cdot \frac{t_1 (1 - r^3)}{1 - r}$$

$$\therefore (1 - r^6) = 9. (1 - r^3)$$

either $1 - r^3 = 0$, i. e. $r^3 = 1$, i. e. $r = 1$

or $1 + r^3 = 9$ so that $r = 2$

Since $r = 1$ is inadmissible

$\therefore r = 2$

Example 2. The continued product of three numbers in G. P. is 216 and the sum of the products of them in pairs is 156. Fnd the numbers.

Let the numbers be $\frac{a}{r}$, a, ar

Then, $\frac{a}{r} \cdot a \cdot ar = 216$ .... (i)

and $\frac{a}{r} \cdot a + a \cdot ar + ar \cdot \frac{a}{r} = 156$ ....(ii)

from (i), $a^3 = 216$

$\therefore a = 6$

Hence from (ii), $\frac{36}{r} + 36r + 36 = 156$

$\therefore 3r^2 - 10r + 3 = 0$

$\therefore (3r - 1)(r - 3) = 0$

$\therefore r = \frac{1}{3}$ or 3.

If $r = \frac{1}{3}$, the numbers are

18, 6, 2

If $r = 3$, the numbers are

2, 6, 18

6·7. The Geometric Mean :

The terms of a G. P. between any two fixed terms are called the geometric means between those two terms.

If a, b, c, are any three numbers in G. P. then b is the geometric mean (G. M.) between a and c.

Since a, b, c are in G. P. therefore

$$\frac{b}{a} = \frac{c}{b}$$

$$\therefore b^2 = ac$$

$$\therefore b = \sqrt{ac} \quad \text{……………} \quad (4)$$

Thus the geometric mean between any two numbers is equal to the square root of their product.

Example 1. Find the geometric mean between $2\frac{7}{8}$ and $5\frac{3}{4}$

Using the formula (4) of art 6·7,

$$\text{the geometric mean} = \sqrt{\frac{23}{8} \times \frac{23}{4}}$$

$$= \frac{23}{4\sqrt{2}}$$

$$= \frac{23}{8}\sqrt{2}$$

Example 2 : Insert 3 geometric means between $2\frac{1}{4}$ and $\frac{4}{9}$

Let G_1, G_2, G_3 be the required geometric means.

So, $\frac{9}{4}$, G_1, G_2, G_3, $\frac{4}{9}$ are in G. P.

By the formula for the nth term, we have

$$\frac{4}{9} = \frac{9}{4}\,(r)^{5-1}$$

$$\therefore\ r^4 = \frac{16}{81} = \left(\frac{2}{3}\right)^4$$

$$\therefore\ r = \frac{2}{3}$$

$$\text{Hence, } G_1 = \frac{9}{4}\cdot r = \frac{9}{4}\cdot\frac{2}{3} = \frac{3}{2}$$

$$G_2 = G_1\cdot r = \frac{3}{2}\cdot\frac{2}{3} = 1$$

$$G_3 = G_2\cdot r = 1\cdot\frac{2}{3} = \frac{2}{3}$$

Thus the geometric means are $\frac{3}{2}$, 1, $\frac{2}{3}$

6·8. Geometric progression with infinitely many terms :

We have discussed and evolved a formula for the sum of n terms of a geometric progression. Throughout the discussion made, n is necessarily a finite positive integer. We are now interested in finding out if the sum of infinitely many terms of a G. P. can be expressed as a single number, and if this is possible, then finding out the formula giving it.

Let us consider the G. P.

$t_1,\ t_1r,\ t_1r^2,\ t_1r^3,\ \dots\ \dots\ \dots$

We shall consider it under two cases.

Case I. If $|r| > 1$ then magnitude of each term is greater than the preceding term. The G. P. therefore, becomes an increasing G. P. and the sum of infinitely many terms cannot be approximated to a number having a definite value.

If $|r| = 1$, all the terms are equal and the sum cannot again be obtained as a number.

Case II. If $|r| < 1$, then the terms become smaller and smaller in magnitude. So, if the number of terms n is inconceivably large then the whole sum tends to a definite number. For example, consider infinitely many terms of the G. P.

$$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots\dots\dots \frac{1}{3^n} \dots\dots\dots$$

We have, from the formula for the sum to n terms,

$$\begin{aligned} S_n &= \frac{t_1}{1-r} - \frac{t_1 r^n}{1-r} \\ &= \frac{\frac{1}{3}}{1-\frac{1}{3}} - \frac{\frac{1}{3}(\frac{1}{3})^n}{1-\frac{1}{3}} \\ &= \tfrac{1}{2} - \tfrac{1}{2}(\tfrac{1}{3})^n \\ &= \tfrac{1}{2}\{1-(\tfrac{1}{3})^n\} \end{aligned}$$

As n gets larger, $(\frac{1}{3})^n$ becomes smaller and smaller. In fact, $(\frac{1}{3})^n$ can be made as small as we please by choosing n sufficiently large. So, when n becomes infinitely large, $(\frac{1}{3})^n$ becomes negligibly small and can be taken to be zero, so that the sum of infinitely many terms of this G.P. becomes $\frac{1}{2}$. This is written as $\underset{n \to \infty}{\text{Limit}}\ S_n = \frac{1}{2}$

The same process holds for the sum of infinitely many terms of any geometric series in which $|r| < 1$.

Consider the expression for the sum of n terms of the general G. P.

$$S_n = \frac{t_1}{1-r} - \frac{t_1 r^n}{1-r}$$

If n is increased indefinitely, r^n decreases indefinitely, t_1 and r remaining constant, so that $\frac{t_1 r^n}{1-r}$ decreases.

Ultimately, as n becomes infinitely large we have, as above,

$$\underset{n \to \infty}{\text{Limit}} \frac{t_1 r^n}{1-r} = 0$$

So that the sum becomes

$$S = \lim_{n \to \infty} S_n = \frac{t_1}{1-r} \; ; |r| < 1 \ldots \ldots (5)$$

Example 1 : Find the sum of infinite terms of the G. P. 16, −4, 1...

$$\text{Here, } t_1 = 16, \; r = -\frac{4}{16} = -\frac{1}{4}$$

Since $|r| = \frac{1}{4} < 1$, therefore the sum of infinite terms of G. P. exists. By formula (5), of this chapter, the sum is given by

$$S = \frac{t_1}{1-r}$$

$$= \frac{16}{1-\left(-\frac{1}{4}\right)}$$

$$= \frac{64}{5}$$

Example 2. Convert the repeating decimal 6·272727....into an equivalent common fraction.

The given number $6\cdot\dot{2}\dot{7}$ can be represented as the number 6 added to an infinite G. P. For,

$$6\cdot\dot{2}\dot{7} = 6 + \cdot 27 + \cdot 0027 + \cdot 000027 + \ldots\ldots$$

$$= 6 + \left[\frac{27}{100} + \frac{27}{10000} + \frac{27}{1000000} + \ldots\right]$$

$$= 6 + \left[\frac{27}{10^2} + \frac{27}{10^4} + \frac{27}{10^6} + \ldots \ldots\right]$$

The series written within the square brackets is an infinite geometrie progression in which $t_1 = \frac{27}{10^2}$

$$\text{and } r = \frac{27/10^4}{27/10^2} = \frac{1}{10^2}; \text{ so that } |r| < 1$$

$$\text{Therefore, } 6\cdot\dot{2}\dot{7} = 6 + \frac{\frac{27}{10^2}}{1-\frac{1}{10^2}}$$

$$= 6 + \frac{27}{99}$$

$$= \frac{69}{11}$$

We therefore observe that every recurring decimal is a rational number since each of them can be expressed as a common fraction.

Example 3 : A ball, dropped from a height of 48 feet, rebounds one third of the distance it falls. How far will it travel before coming to rest if it keeps falling and rebounding in this way?

After the first impact with the floor, the ball describes each time twice the distance it rises in order to come in contact with the floor again. The sum of the distances described is given by

$$S = 48 + 2\left[48\cdot\frac{1}{3} + 48\cdot\left(\frac{1}{3}\right)^2 + 48\cdot\left(\frac{1}{3}\right)^3 + \dots\right]$$

$$= 2\left[48 + 48\cdot\frac{1}{3} + 48\left(\frac{1}{3}\right)^2 + \dots\right] - 48$$

$$= 2\cdot\frac{48}{1-\frac{1}{3}} - 48, \text{ using formula (5) of this chapter.}$$

$$= 96.$$

6·9 Miscellaneous Illustrative Examples :

Example 1 : The sum of three numbers in G. P. is 38 and their product is 1728. Find them.

Let the three numbers be

$$\frac{a}{r},\ a,\ ar.$$

Therefore, $\frac{a}{r} + a + ar = 38$ ...... (i)

and $\frac{a}{r} \times a \times ar = 1728$ ......(ii)

from (ii), $a^3 = 1728$

$\therefore a = 12$

Substituting in (i),

$$\frac{12}{r} + 12 + 12r = 38$$

$$\therefore 6r^2 - 13r + 6 = 0$$

$$\therefore (3r-2)(2r-3) = 0$$

$$\therefore r = \frac{2}{3} \text{ or } \frac{3}{2}.$$

If $r = \frac{2}{3}$, the numbers are 18, 12, 8.

If $r = \frac{3}{2}$, the numbers are 8, 12, 18.

Example 2 : Sum to n terms the series 4 + 44 + 444 + ...

Let the sum be denoted by S_n.

$\therefore S_n = 4 + 44 + 444 + $ ......continued to n terms.

$= 4\ [1 + 11 + 111$ ......continued to n terms]

$= \frac{4}{9}\ [9 + 99 + 999 +$ ......continued to n terms]

$= \frac{4}{9}\ [(10-1) + (10^2-1) + (10^3-1)$...contd. to n terms]

$= \frac{4}{9}\ [\{10 + 10^2 + 10^3 +$ ......contd. to n terms$\} - n]$

$$= \frac{4}{9}\left[\frac{10(10^n-1)}{10-1} - n\right]$$

$$= \frac{4}{9}\left[\frac{10}{9}\,(10^n-1)-n\right]$$

$$= \frac{40}{81}\left[(10^n-1)-\frac{4}{9}\,n\right].$$

Example 3 : If $x < 1$, sum the series,

$1 + 3x + 5x^2 + 7x^3 + ...$contd. to infinity.

Let $S = 1 + 3x + 5x^2 + 7x^3 + ...$contd. to ∞

$\therefore\ x \cdot S = x + 3x^2 + 5x^3 + ...$contd. to ∞

$\therefore\ S - x \cdot S = 1 + 2x + 2x^2 + 2x^3 + ...$contd. to ∞

$\therefore\ S\,(1-x) = 1 + 2\,\{x + x^2 + x^3 + ...$contd. to $\infty\}$

$$= 1 + \frac{2 \cdot x}{1-x}$$

$$= \frac{1+x}{1-x}$$

$$\therefore\quad S = \frac{1+x}{(1-x)^2}$$

The above series is typical of the series in which each term can be expressed as the product of two factors such that the first factors of all the terms constitute an A. P. and the second factors of all the terms constitute a G. P. Such a series is sometimes called an "Arithmetico-geometric series."

EXERCISE

1. Sketch the graph for each of the following functions.

   (i) 3^x, (ii) $\left(\frac{1}{2}\right)^{-x}$, (iii) $\left(\frac{1}{3}\right)^{x}$

2. Find t_n and S_n of the geometric progression
   $c\,(1+d)$, $c\,(1+d)^2$, $c\,(1+d)^3 \ldots.$

3. Sum the following geometric progressions :

(i) $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots\ldots$ to 9 terms.

(ii) $\sqrt{3}, \frac{1}{\sqrt{3}}, \frac{1}{3\sqrt{3}} \ldots\ldots$ to 7 terms

4. Continue to 3 more terms the series

$$\frac{1}{125}, -\frac{1}{25}, \frac{1}{5} \ldots \ldots \ldots$$

5. Find t_n and S_n of

$1, \cdot 1, \cdot 01, \cdot 001, \ldots \ldots \ldots \ldots$

6. If $\frac{1}{x+y} + \frac{1}{y+z} = \frac{1}{y}$, then show that

x, y, z, are in G. P.

7. Find the geometric means between the following pairs of numbers:

(i) $\frac{4}{9}, 1$; (ii) 4, 9; (iii) 3, 12;

(iv) $\frac{1}{3}, \frac{3}{4}$; (v) $\frac{5}{3}, \frac{1}{15}$.

8. Insert three geometric means between : $\frac{1}{2}$ and $\frac{4}{9}$.

9. Insert two geomteric means between : 1 and $\frac{8}{27}$.

10. Insert four geometric means between : $\frac{1}{2}$ and $\frac{1}{486}$.

11. Insert three geometric means between : $\frac{1}{3}$ and $\frac{27}{16}$.

12. Find the value of y so that $2y + 5, 4y + 3, 8y - 7$ will form a geometrical progression.

13. In a geometric progression, the third term is $\frac{25}{4}$ and the 7th term is $\frac{4}{25}$. Find the 5th and the 8th terms.

14. The 4th term of a geometric progression is 2, and the 6th term is 8. Find the first three terms.

15. In the geometric progression $-\frac{1}{2}, \frac{1}{3}, -\frac{2}{9}, \frac{4}{27} \dots\dots$ which term is $\frac{2^8}{3^9}$?

16. Prove that for two unequal positive numbers a and b, the arithmetic mean is greater than the geometric mean.

    [Hint : $(\sqrt{a}-\sqrt{b})^2 > 0$]

17. If you are to choose between two jobs in one of which the salary is Rs. 1000 per month and in the other it is on daily basis such that the first day it is Re. 1, second day Rs 2, third day Rs. 4 and so on, each day the salary being double that of the previous day, which job will fetch you more money, if you wish to work only for (i) a week, (ii) a fortnight, (iii) a month ?

18. If a and b be the first two terms of a G.P. in which there are n terms and l be the last term, then show that $b = \frac{a\,(S_n-a)}{(S_n-l)}$.

19. Prove the formula $S_n = \frac{t_1\,(1-r^n)}{1-r}$ of Art. 6·6 by the method of induction.

20. If a, b, c, d are in G. P. show that $\frac{ab-cd}{b^2-c^2} = \frac{a+c}{b}$.

21. Show that in any G. P. t_n, t_{2n}, t_{3n} themsleves form a G. P.

22. If a, b, c, d are in G. P. then show that

    $(b-c)^2 + (c-a)^2 + (d-b)^2 = (a-d)^2$

23. If a, b, c, d are in G. P., then prove that
$a^2 - b^2, b^2 - c^2, c^2 - d^2$ are in G.P.

24. If S be the sum of an infinite number of terms of a G. P. and σ the sum of the squares of the terms, prove that

$$a = \frac{2S\sigma}{S^2 + \sigma} \text{ aud } r = \frac{S^2 - \sigma}{S^2 + \sigma}$$

25. If S_1, S_2, S_3 be the sums of n, 2n, 3n terms respectively of a G. P., prove that $S_1(S_3 - S_2) = (S_2 - S_1)^2$.

26. Sum to n terms:
(i) $3 + 33 + 333 + \ldots\ldots\ldots$
(ii) $\cdot 7 + \cdot 77 + \cdot 777 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

27. Find the sum of n terms of the series

$$1 + \left(1 + \frac{1}{2}\right) + \left(1 + \frac{1}{2} + \frac{1}{2^2}\right) + \cdots\ \cdots\ \cdots\ \cdots$$

$$\left[\text{Hint} : t_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\cdots + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}\right].$$

28. The sum of an infinite geometric progression is $\frac{7}{2}$ and the first term is 3. What is the common ratio?

29. The middle third of a line 1 ft. long is erased. From each of the two remaining thirds, the middle thirds are erased. From each of the 4 remaining parts, the middle thirds are erased. If this erasing continues for 10 times, what will be the sum of the lengths of the remaining line segments?
[Hint: the required sum = 1 — (sum of the lengths erased). Note that the lengths of the portions erased form a G. P.]

30. Sum the series
$1 + 3x + 6x^2 + 10x^3 + \ldots\ldots\ldots$ contd. to ∞ . $(x < 1)$

31. If $y = x + x^2 + x^3 + \ldots$ contd. to ∞ where x is positive and less than 1, prove that $x = \frac{y}{1 + y}$.

32. Convert the repeating decimal $3 \cdot 242424\ldots$ into its equivalent common fraction.

33. Express as a proper fraction the recurring decimal $0 \cdot 555\ldots$.

34. If the p^{th}, q^{th}, r^{th} terms of a G. P. are themselves in G. P. then prove that p, q, r are in A.P.

35. In a G. P. if S, P and R denote respectively the sum, the product and the sum of the reciprocals of n terms, prove that

$$P^2 = \left(\frac{S}{R}\right)^n$$

36. If a, b, c are in A. P. and a, x, b and b, y, c are in G. P. show that x^2, b^2, y^2 are in A. P.

ANSWERS

2. $t_n = c\,(1 + d)^n$, $S_n = \frac{c}{d}\,(1 + d)\,\{(1 + d)^n - 1\}$.

3. $\frac{171}{256}$, $\frac{1093\sqrt{3}}{729}$.

4. $-1, 5, -25$.

5. $\frac{1}{10^{n-1}}$, $\frac{10^n - 1}{9 \cdot 10^{n-1}}$.

7. (i) $\frac{2}{3}$, (ii) 6, (iii) 6, (iv) $\frac{1}{2}$, (v) $\frac{1}{3}$.

8. $\frac{3}{2}$, 1, $\frac{2}{3}$.

9. $\frac{2}{3}$, $\frac{4}{9}$.

10. $\frac{1}{6}$, $\frac{1}{18}$, $\frac{1}{54}$, $\frac{1}{162}$.

11. $\frac{1}{2}$, $\frac{3}{4}$, $\frac{9}{8}$.

12. 22.

13. 1, $\frac{8}{125}$.

14. $\frac{1}{4}$, $\frac{1}{2}$, 1.

15. 10.

17. Ist, 2nd, 2nd.

26. (i) $\frac{10(10^n-1)}{27} - \frac{n}{3}$.

(ii) $\frac{7n}{9} - \frac{7}{81}\left(1-\frac{1}{10^n}\right)$.

27. $2n + \frac{1}{2^{n-1}} - 2$.

28. $\frac{1}{7}$.

29. $\left(\frac{2}{3}\right)^{10}$.

30. $\frac{1}{(1-x)^3}$

32. $\frac{8}{33}$,

33. $\frac{5}{9}$.

6·10. The Logarithmic Function :

In art. 6·2 we discussed the exponential function $y = a^x$, $x \in R$ and $a > o$. In this function, the values of the exponent x constitute the domain and the values of y (that is a^x) constitute the range. Let us now consider the inverse function in which the domain and the range are interchanged.

Consider, therefore, the function $x = a^y$, $y \in R$ and $a > 0$, $a \neq 1$

For reasons which will be obvious shortly we put the restriction $a \neq 1$. on $a > 0$ in the exponential function $x = a^y$, the domain being the set of real values of the exponent y and the range the values of x (that is a^y). If now the domain and the range are interchanged that is the set of values of x (that is a^y) is made the domain and the set of values of the exponent y is considered to be the range, then the correspondence obtained by the rule $x = a^y$ will be a function. Because, for any positive number x and any positive number $a \neq 1$, there is one and only one real value of y which satisfies the equation $a^y = x$. [Would this have been true if we had allowed 'a' to be equal to 1?]

So, given a fixed value of 'a,' any object x will correspond to one and only one real number y, and thus, the association $x = a^y$ with values of x as domain and values of the exponent y as the range is a function which has been obtained by interchanging the domain and the range of the exponential function.

The equation $a^y = x$ can be solved for y and we can express y in terms of x and a. We then call y the logarithm of x to the base a and write

$$y = \log_a x$$

We thus define a new function as follows:

Definition : The function $F : x \rightarrow \log_a x$ where $x > 0$, $a > 0$, and $a \neq 1$ is called the **logarithmic function**.

It is clear from the discussions made above that the exponential function $f : x \rightarrow a^x$ and the logarithmic function $F : x \rightarrow \log_a x$ are inverse functions in the sense that the domain of one is the range of the other and vice-versa.

Also, algebraically, the two expressions $x = a^y$ and $y = \log_a x$ are equivalent. Thus, writing $9 = 3^2$ is same as saying $2 = \log_3 9$.

Saying $\frac{1}{16} = \left(\frac{1}{2}\right)^4$ is same as saying $4 = \log_{\frac{1}{2}}\left(\frac{1}{16}\right)$

Saying $\log_5 125 = 3$ is same as saying $5^3 = 125$

Saying $\log_{\frac{1}{3}}\left(\frac{1}{27}\right) = 3$ is same as saying $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$

This eqivalence makes evident many of the properties of logarithms which we shall be considering later. Let us discuss the graph of the logarithmic function before going over to the algebraic properties of the logarithms.

6·11. Graph of the logarithmic function:

Let $F : x \rightarrow \log_a x$, $x > 0$, $a > 0$ and $a \neq 1$.

then denoting as usual the image by y, we have, $y = \log_a x$,

which is the same as $x = a^y$, an expression exactly the same as the exponential function with the positions of x and y interchanged. In drawing the graph of the logarithmic function $y = \log_a x$

we have actually to draw the graph for the function $x = a^y$ (remember $a \neq 1$) and therefore we expect the graph to be a curve which behaves with respect to the x-axis as the exponential curve does with respect to the y-axis and vice-versa. In other words, if the axes of x and y were interchanged, the exponential curve becomes the logarithmic curve and of course, conversely.

The only basic difference in drawing the curves of the exponential and logarithmic functions is that for the former, a curve corresponded to the case $a = 1$; whereas for the latter since it is defined under the restriction $a \neq 1$, no real curve can be drawn when $a = 1$.

We therefore, consider the graph of the logarithmic function under the following two cases :

Case I : $a > 1$

Let $a = 2$

The logarithmic function is $F : x \rightarrow \log_2 x$

that is, $y = \log_2 x$

that is, $x = 2^y$.

We tabulate a few ordered pairs by taking some chosen values of x (i.e. object) and obtaining the corresponding values of y (i. e. image). observing the variation of y with the variation in x, the following general rules about the graph can be formed.

x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
y	-4	-3	-2	-1	0	1	2	3	4

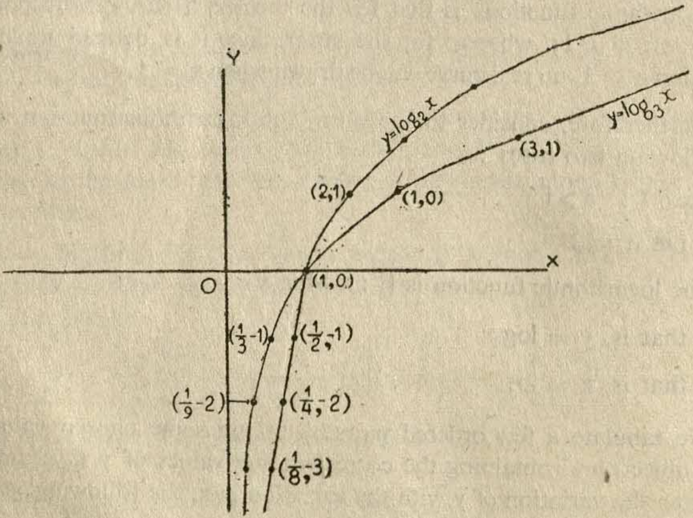
(1) The graph lies entirely to the right of the y axis (since by definition, $x > 0$) that is, in the I^{st} and 4^{th} quadrants. It does not exist in 2^{nd} or 3^{rd} quadrant.

(2) As x increases, so does y. The function is therefore, an increasing function. For values of x greater than 1, y is positive while for fractional values of x less than 1, y is negative. So, for all values of $x > 1$ the curve lies below the x-axis. It crosses the axis of x at (1,0).

(3) The function is not defined for –ve values of x.

(4) The function has only one zero and it is at $x = 1$

(5) Whatever be the value of a, the point (1, 0) always lies on the exponential curve.



(6) Since only x^1 is equal to x, therefore $\log_x x = 1$ i.e. logarithm of any number to the same base is always equal to 1. Thus y will be equal to one when the value of x is the same as the value of the base a. The property $\log_x x = 1$ will be proved algebraically later in this chapter.

(7) The curve never meets the y axis since 2^y can never be zero.

The geometrical interpretation of the logarithmic function for $a = 2$ and $a = 3$ are shown below.

Note that the y axis is an asymptote to the curve at its negative side.

Case II : $a < 1$

If $a = \frac{1}{2}$, then the values of y corresponding to those of x are given by

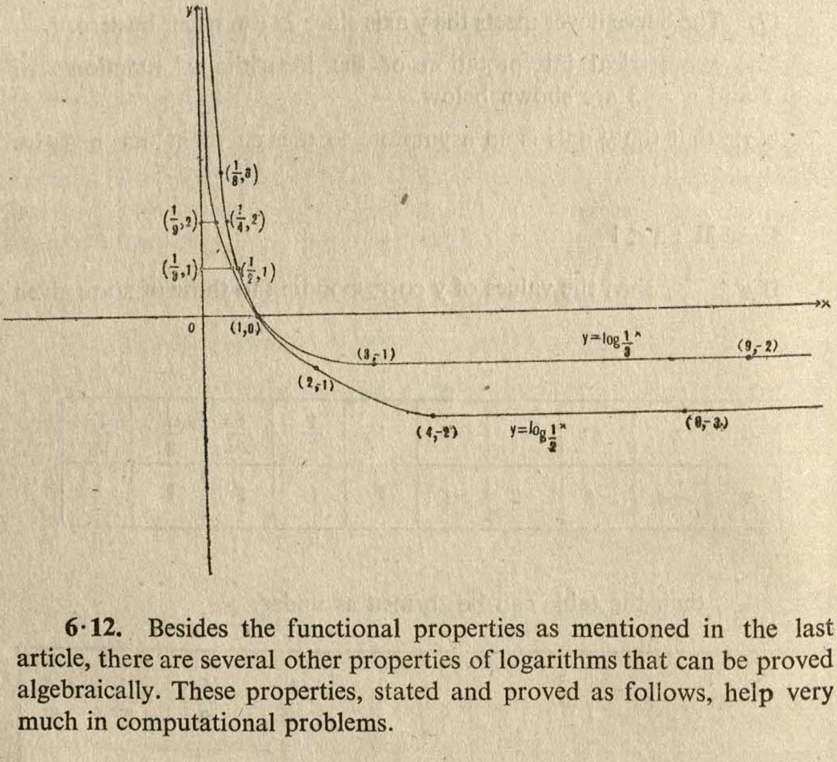
x	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
y	−4.	−3	−2	−1	0	1	2	3	4

If $a = \frac{1}{3}$, then the table can be formed as under.

X	81	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{81}$
y	−4	−3	−2	−1	0	1	2	3	4

All observations except (2) are exactly the same in this case as in case I. The observation (2) changes because in this case, while the function is an increasing function, the curve lies to the −ve side of the y-axis for values of x greater than 1 and it lies to the +ve side of the y-axis for +ve values of x less than 1. So for all values of x $>$ 1, the curve lies below the x-axis and for all values of x $<$ 1, the curve lies above the x-axis.

The graph can be drawn as shown in the figure. Note that for each a $<$ 1 the y-axis is an asymptote to the curve at its positive side.



6·12. Besides the functional properties as mentioned in the last article, there are several other properties of logarithms that can be proved algebraically. These properties, stated and proved as follows, help very much in computational problems.

Property 1: The logarithm of 1 to any base (except base 1) is zero

Let $\log_a 1 = x, \quad a > 0, a \neq 1.$

Then $a^x = 1$

So that $x = 0$

Thus $\log_a 1 = 0$ where a is any +ve number.

Property 2 : The logarithm of any number to the same base is 1.

Let $\log_a a = x, \quad a > 0$

Then, $a^x = a$

So that $x = 1$

Therefore $\log_a a = 1$

Logarithm of zero is undefined.

Property 3 : The logarithm of the product of two quantities is equal to the logarithm of the first quantity added to the logarithm of the second.

That is, $\log_a uv = \log_a u + \log_a v$, $a > 0$, where u and v are any two +ve numbers.

Let $\log_a u = x$ and $\log_a v = y$

$\therefore \quad u = a^x$ and $v = a^y$

$\therefore \quad uv = a^x \cdot a^y = a^{x+y}$

$\therefore \quad \log_a uv = x + y$

$\therefore \quad \log_a uv = \log_a u + \log_a v$

Property 4 : The logarithm of the quotient of two quantities is equal to the logarithm of the numerator minus the logarithm of the denominator.

That is, $\log_a \frac{u}{v} = \log_a u - \log_a v; \ a>0.$

where u and v are any two +ve numbers.

Let $\log_a u = x$ and $\log_a v = y$

$\therefore \quad u = a^x$ and $v = a^y$

$$\therefore \quad \frac{u}{v} = \frac{a^x}{a^y} = a^{x-y}$$

$$\therefore \quad \log_a \left(\frac{u}{v}\right) = x - y$$

$$\therefore \quad \log_a \left(\frac{u}{v}\right) = \log_a u - \log_a v$$

Property 5 : The logarithm of a power of a quantity is equal to the power multiplied by the logarithm of the quantity.

That is, $\log_a u^p = p \log_a u;\ a > 0.$
where u is any positive quantity and p is any real number.

$$\text{Let } \log_a u = x$$
$$\therefore\ u = a^x$$
$$\therefore\ u^p = a^{xp}$$
$$\therefore\ \log_a(u^p) = xp$$
$$\therefore\ \log_a(u^p) = p \log_a u$$

Property 6 : The logarithm of any quantity to a given base equals the product of the logarithm of the same quantity to any new base and the logarithm of the new base with respect to the given base.

That is, $\log_a m = \log_b m \times \log_a b,\ a > 0, \neq 1;\ b > 0, \neq 1$
m is any +ve number.

Let $\log_a m = x$ and $\log_b m = y$

$$\therefore\ m = a^x \text{ and } m = b^y$$
$$\therefore\ a^x = b^y$$
$$\therefore\ a^{x/y} = b$$
$$\therefore\ \log_a b = \frac{x}{y}$$
$$\therefore\ x = y \log_a b$$
$$\therefore\ \log_a m = \log_b m \times \log_a b$$

If m = a, then we have

$$\log_a a = \log_b a \times \log_a b$$
$$\therefore\ 1 = \log_b a \times \log_a b$$
$$\therefore\ \log_b a = \frac{1}{\log_a b}$$

6·13. Examples :

(1) Express as a single logarithm

$\log_b x + 5 \log_b (x + 2) - 3 \log_b x^7$

By Property 5,

$$\log_b x + 5 \log_b (x + 2) - 3 \log_b x^7$$

$$= \log_b x + \log_b (x + 2)^5 - \log_b (x^{21})$$

$$= \log_b x (x + 2)^5 - \log_b x^{21} \qquad \text{by Prop. 3}$$

$$= \log_b \frac{x (x + 2)^5}{x^{21}} \qquad \text{by Prop. 4}$$

(2) Simplify :

$$\log_{10} \frac{(54\cdot3)^4 (67)}{(82\cdot7)(23\cdot4)}$$

By properties 3 and 4, we have

$$\log_{10} \frac{(54\cdot3)^4 (67)}{(82\cdot7)(23\cdot4)}$$

$$= \log_{10} (54\cdot3)^4 + \log_{10} (67) - \log_{10} (82\cdot7) - \log_{10} (23\cdot4)$$

$$= 4 \log_{10} (54\cdot3) + \log_{10} 67 - \log_{10} (82\cdot7) - \log_{10} (23\cdot4) \qquad \text{(by prop. 5)}$$

$$= 4 \log_{10} \frac{543}{10} + \log_{10} 67 - \log \frac{827}{10} - \log_{10} \frac{234}{10}$$

$$= 4 [\log_{10} 543 - \log_{10} 10] + \log_{10} 67 - [\log_{10} 827 - \log_{10} 10] - [\log_{10} 234 - \log_{10} 10] \qquad \text{(by prop. 5)}$$

$$= 4 \log_{10} 543 - 4 + \log_{10} 67 - \log_{10} 827 + 1 - \log_{10} 234 + 1 \qquad \text{(by prop. 2).}$$

$$= 4 \log_{10} 543 + \log_{10} 67 - \log_{10} 827 - \log_{10} 234 - 2.$$

(3) If $x = \log_{\sqrt{6}} 36$ find the value of x.

Here $\log_{\sqrt{6}} 36 = x$

$\therefore (\sqrt{6})^x = 36$

$\therefore (6)^{x/2} = 36 = 6^2$

$\therefore \frac{x}{2} = 2$

$\therefore x = 4$

EXERCISE

1. Sketch the graphs of the following functions.

   (i) $y = \log_4 x$, (ii) $y = \log_5 x$

   (iii) $y = \log_{\frac{1}{4}} x$, (iv) $y = \log_{\frac{1}{5}} x$.

2. Express the following equations in terms of logarithms

   (i) $5^2 = 25$, (ii) $2^{-10} = \frac{1}{1024}$, (iii) $\left(\frac{2}{3}\right)^{-4} = \frac{81}{16}$,

   (iv) $10^2 = 100$, (v) $(21)^0 = 1$

   (vi) $10^{-2} = \cdot 01$, (vii) $10^{-3} = \cdot 001$, (viii) $t^m = k$,

3. Express in exponent notations the following equations.

   (i) $\log_{\frac{1}{2}} 32 = -5$, (ii) $\log_{64} (8) = \frac{1}{2}$.

   (iii) $\log_{27} 3 = \frac{1}{3}$, (iv) $\log_4 64 = 3$.

   (v) $\log_{36} 1 = 0$,

4. Find the value of x in each of the following equations and express it free from logarithm.

   (i) $\log_2 x = 3$ (ii) $\log_5 x = -2$,

   (iii) $\log_x 32 = -\frac{5}{6}$, (iv) $\log_x 81 = 2$,

   (v) $\log_x \frac{1}{8} = -\frac{3}{4}$, (vi) $\log_3 81 = x$.

   (vii) $\log_2 128 = x$, (viii) $\log_7 \left(\frac{1}{7}\right) = x$.

5. Assuming the law for the logarithm of product, prove the following law by the method of induction.

$$\log_a m^n = n \log_a m\ .$$

Express each of the following as a single logarithm.

6. $4 \log_m a - 3 \log_m b$

7. $2 \log_m x + 5 \log_m y + \log_m z$

8. $\frac{1}{3} \log_a m - \frac{2}{3} \log_a n$

9. $5 (\log_a x + \log_a z) - \log_a 2b$

10. $\frac{1}{2} \log_x t - 2\left(\frac{1}{3} \log_x m + \frac{1}{5} \log_x n\right)$

11. $3 (4 \log_m a - 5 \log_m b) + (7 \log_m c - 3 \log_m d)$

Simplify

12. $\log_a (23 \times 10{\cdot}5)$

13. $\log_a \frac{64 \times 125}{27}$

14. $\log_{10} \frac{(54)^{\frac{1}{3}} (81)^{\frac{3}{4}}}{(20)^2 (30)^3}$

15. $\log_{10} \frac{1}{\sqrt[5]{71 \times 220}}$

16. $\log_{10} \frac{\sqrt[3]{729 \times 64}}{\sqrt[4]{49 \times 25}}\ .$

17. Prove that

$$3 \log_7 \frac{4}{35} + 2 \log_7 \frac{5}{49} - \log_7 \frac{20}{343} - 2 \log_7 \frac{4}{5} + \frac{5}{2} \log_7 49 = 1$$

18. If x, y, z are in G. P., prove that $\log_a x$, $\log_a y$, $\log_a z$ are in A. P.

ANSWERS

2. (i) $\log_5 25 = 2$, (ii) $\log_2 \left(\frac{1}{1024} \right) = -10$,

(ii) $\log_{\frac{2}{3}} \frac{81}{16} = -4$ (iv) $\log_{10} 100 = 2$,

(v) $\log_{21} 1 = 0$, (vi) $\log_{10} \cdot 01 = -2$,

(vii) $\log_{10} \cdot 001 = -3$, (viii) $\log_t k = m$.

3. (i) $\left(\frac{1}{2} \right)^{-5} = 32$, (ii) $(64)^{\frac{1}{2}} = 8$,

(iii) $(27)^{\frac{1}{3}} = 3$, (iv) $4^3 = 64$, (v) $36^{\circ} = 1$.

4. (i) 8, (ii) $\frac{1}{25}$, (iii) $\frac{1}{64}$, (iv) 9,

(v) 16, (vi) 4, (vii) 7, (viii) -1.

6. $\log_m \frac{a^4}{b^3}$

7. $\log_m (x^2 y^5 z)$

8. $\log_a \left(\frac{m}{n^2} \right)^{1/3}$

9. $\log_a \frac{x^5 z^5}{2b}$

10. $\log_x \frac{t^{1/2}}{m^{\frac{2}{3}} n^{2/5}}$

11. $\log \frac{a^{12} c7}{b^{15} d^3}$

12. $\log_a 23 + \log_a 21 - \log_a 2$

13. $6 \log_a 2 + 3 \log_a 5 + 3 \log_a 3$

14. $\log_{10} 3 - \frac{5}{3} \log_{10} 2 - 5$

15. $\frac{1}{5} \left(- \log_{10} 71 - \log_{10} 220 \right)$

16. $2 (\log_{10} 3 + \log_{10} 2) - \frac{1}{2} (\log_{10} 7 + \log_{10} 5)$

CHAPTER VII

Limits and Derivatives

7·1. Example 1. Let f(x) be a function of x defined by

$$f(x) = \frac{x^2 - 3x}{x - 3} \quad ; \; x \neq 3, x \in R.$$

Clearly, the function is not defined at x = 3 and hence f(3) does not exist. But we can find a number to which the values of f(x) approximate for the values as near to x = 3 as possible, although at x = 3 itself, the value does not exist. Consider the following table.

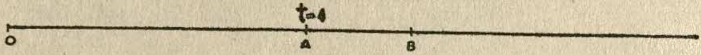
X	2.8	2.85	2.9	2.99	2.999	2.9999	......	......	......	......	3.0001	3.001	3.01	3.1	3.15	3.2
$f(x)$	5.6	5.70	5.8	5.98	5.998	5.9998	......	......	......	......	6.0002	6.002	6.02	6.2	6.3	6.4

It is obvious from this table that as x approaches the value 3, f(x) assumes values nearer and nearer to 6. So we conclude that the value of f(x) must be very nearly equal to 6 for x close to x = 3. We term this as "The value of f(x) tends to become 6 as x tends to 3", and write it as

$$\lim_{x \to 3} f(x) = 6.$$

Thus, though we do not know the value of f(x) at x = 3, we can guess it to an accurate approximation by observing values of $f(x)$ for a number of values of x in a close surrounding of $x = 3$.

Example 2 : A particle moves along a straight line under the law $s = 3t^2$ where t is the time elapsed from its starting position and s is the distance described in time t. Find the velocity of the body at the instant $t = 4$



If the velocity of a body is uniform, it is given by the distance travelled during a certain length of time divided by the time in seconds. If, however, the velocity is not uniform, as in this problem, then to find the velocity at any point A (at the instant $t = 4$) we consider the time taken by the body to travel a distance AB where B is some other point near A on the line of motion. The distance AB divided by the time taken to traverse this distance is the average velocity over AB. Since the velocity of the body is changing at every point, the average velocity over AB will not be equal to the actual velocity at A. But if B is taken very near to the point A, that is, if the distance AB is taken very small then the average velocity over AB will approximate to the velocity at A and the nearer B is to A the better will the approximation be. In other words, as B approaches A, the approximate value approaches the value of the velocity at A.

Consider therefore a point B which is arrived at by the particle at time $(4 + h)$ seconds $(h > 0)$,

Since the distance is given by $s = 3t^2$,

Distance OA $= 3 \cdot 4^2 = 48$ ft.

and Distance OB $= 3 \cdot (4 + h)^2 = 48 + 24h + 3h^2$ ft.

$$\therefore \text{Average velocity over AB} = \frac{\text{Distance AB}}{\text{time taken to traverse AB}}$$

$$= \frac{48 + 24h + 3h^2 - 48}{(4 + h) - 4}$$

$$= \frac{24h + 3h^2}{h}$$

$$= 24 + 3h \ \text{ft/sec}^2.$$

As explained above, this average velocity will be approximately equal to the velocity at A provided B is quite near to A. Let us now take B nearer and nearer to A, that is, take h smaller and smaller, find out in each case the average velocity over AB and approximate out the velocity of the particle at A.

h in secs	AB in ft	Average velocity over AB in ft/sec^2 $\left(\text{given by } \frac{AB}{h}\right)$
0·1	2·43	24·3
0·01	·2403	24·03
0·001	·024003	24·003
0·0001	·00240003	24·0003

Thus, as h becomes smaller and smaller, AB also becomes smaller and smaller, (meaning that B approaches A). From the table above, as B approaches A, the average velocity over AB differs less and less from the value 24 ft/sec. We therefore conclude that the velocity at A is 24 ft/sec. This is also expressed as "the limit of the velocity as time t tends to become 4 secs, is 24 ft/sec." and write it as $\lim\limits_{t \to 4}$ (velocity) = 24 ft./sec.

The velocity of the particle at any other instant of its motion may similarly be found.

7·2. The two examples above show that, if any expression in terms of a variable is undefined for a certain value of the variable then the number to which the values of the expression approximate for that value of the variable can be obtained by considering a great number of values as near to the required value of the variable as possible. In ex. 1 of the last article, we required the value of f(x) at $x = 3$. So we chose a number of values of x as near $x = 3$ as possible and watched what sort of corresponding values we get for f(x). Since the values of f(x) we got, clustered around 6, we concluded that f(x) approximates to the value 6 as x approaches the value 3.

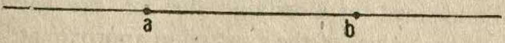
In ex. 2. of the same art. the values of t are chosen in the vicinity of 4 to crowd round $t = 4$ and correspondingly we find that the values of the velocity adhere around $v = 24$. In fact, closer t is to 4, nearer v is to 24. We therefore conclude that the velocity of the particle is 24ft/sec when $t = 4$ sec.

*7·3. **Interval:** We know that the set of real numbers is equivalent to the set of points in a straight line. Each real number can be represented by a point in a line and each point of a line represents a real number. If P and Q are any two fixed points on a straight line P lying to the left of Q, then the set of all points between P and Q forms a subset of the set R of all real numbers.

Let P and Q represent the real numbers a and b respectively, a preceding b. The set of all points between P and Q is equivalent to the set of all real numbers between a and b, and this set as mentioned before, is a subset of R. Such a subset (of all real numbers lying between two fixed real numbers) of R is called an "*interval.*"

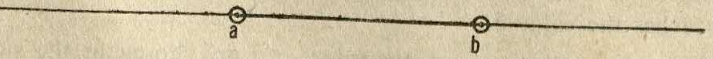
The question now arises as to whether the numbers a and b should themselves belong to the interval. To provide an answer, we make the following classification:

(i) If the end points a and b belong to the interval then it is called a closed interval and is denoted as [a, b]; by putting a and b within a box bracket.



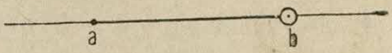
$\therefore [a, b] = \{x/a \leqslant x \leqslant b\}$

(ii) If the end points a and b do not belong to the interval then it is called an open interval and is written as (a, b), by putting a and b within a round bracket. $\therefore (a, b) = \{x \mid a < x < b\}$



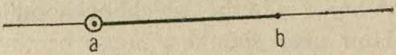
(iii) If a belongs to the interval but b does not, then it is called a semiclosed (or semiopen) interval and is denoted as [a, b)

$\therefore \; [a, b) = \{x \mid a \leqslant x < b\}$



Another semiclosed interval is formed if a does not belong to the interval but b does. It is denoted by (a, b]

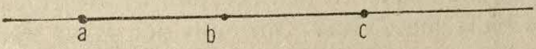
$\therefore \; (a, b] = \{x \mid a < x \leqslant b\}$



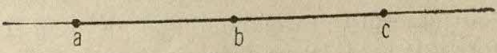
Since an interval is nothing but a set, every property of a set and operations on sets are applicable to intervals. Also, since there are an infinite number of points between any two fixed points on a straight line, every interval is an infinite set.

Illustrations :

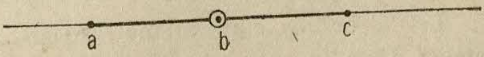
1. $[a, b] \cup [b, c] = [a, c]$



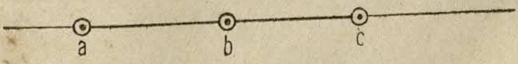
2. $[a, b] \cup (b, c] = [a, c] = [a, b) \cup [b, c]$



3. $[a, b) \cup (b, c] = [a, c] \sim \{b\}$



4. $(a, b) \cup (b, c) = (a, c) \sim \{b\}$

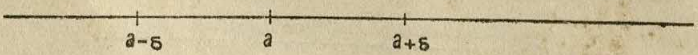


5. $[a, b] \cap [b, c] = \{b\}$
6. $[a, b) \cap [b, c] = \emptyset = [a, b] \cap (b, c]$
7. $(a, b) \cap [b, c] = \emptyset = (a, b) \cap [b, c] = (a, b] \cap (b, c)$
8. $(a, b) \cap (b, c) = \emptyset = [a, b) \cap (b, c] = [a, b) \cap (b, c)$ etc.

*7·4. Neighbourhood :

In everyday life the word "neighbour" means a person who stays close to you. "Neighbourhood" gives us an idea of closeness, nearness. A house is in the "neighbourhood" of your house if it is near your house. How near should a house be to your house so that it may be said to be in your "neighbourhood"? Generally, there is no hard and fast rule to be applied to the distance between two houses before one can be said to be in the neighbourhood of the other; although intuitively we always understand any sentence using the word "neighbourhood". Take another instance. We frequently use phrases like "India and Ceylon are neighbouring countries"; "China is a neighbour of India". Would you call a house to be in your neighbourhood if the distance between this house and yours was separated by the length of the Bay of Bengal that separates India and Ceylon? Or, if the two houses were situated at a distance equal to that occupied by the Himalayan ranges that separate India and China? Obviously not. What we mean is that the word "neighbourhood" in ordinary language has a very flexible meaning. While always meaning "closeness", it is used to signify varied degree of closeness from context to context and is intuitively understood depending upon the reference of the sentence it is being used in. In mathematics too the word "neighbourhood" is used to imply "closeness". But before it is used, let us define it rigorously so that when we speak about "neighbourhood" we shall know what exactly we are speaking about.

Definition : Let 'a' be any point in the real line, and δ be any positive real number. Then the open interval $(a - \delta,\ a + \delta)$ is



said to be the "δ - neighbourhood of a" and is written in short as "δ - nbd of a" or, sometimes as $N_{\delta}(a)$.

Referring to the last article, we thus have,

δ-nbd of a $= (a-\delta,\ a+\delta) = \{x \mid a-\delta < x < a+\delta\}$.

Note that : (i) "Neighbourhood" is an open interval. Hence, it is a set of an infinite number of points.

(ii) The end points are situated symmetrically about 'a' on both sides of it but do not themselves belong to the neighbourhood.

(iii) The condition $a-\delta < x < a+\delta$ is equivalent to the condition $|x-a| < \delta$.

For, as $|x-a|$ is the absolute value of $(x-a)$, the quantity $(x-a)$ can be positive or negative (or zero, which is a trivial case since $\delta > 0$.) Now, if $(x-a)$ is +ve then $|x-a| = x-a$ so that
$|x-a| < \delta$ means $(x-a) < \delta$. i. e. $x < a+\delta$....(i)
and if $(x-a)$ is —ve, then $|x-a| = -(x-a)$
so that $|x-a| < \delta$ means $-(x-a) < \delta$ i. e. $x-a > -\delta$

i. e. $x > a-\delta$.........(ii)

combining (i) and (ii),

$|x-a| < \delta$ means $a-\delta < x < a+\delta$.

Conversely, $a-\delta < x < a+\delta$ is a combination of the two conditions

$x < a+\delta$
and $x > a-\delta$.

which mean, together, $|x-a| < \delta$.

So, $a-\delta < x < a+\delta$ is the same as $|x-a| < \delta$.

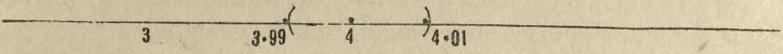
Therefore we write

$$\begin{aligned}\delta\text{—nbd of } a &= (a-\delta,\ a+\delta)\\ &= \{x \mid a-\delta < x < a+\delta\}\\ &= \{x \mid\ |x-a| < \delta\}\end{aligned}$$

Illustrations :

1. Does 3 belong to the (i) $\cdot 01$ nbd of 4?
   (ii) $\cdot 25$ nbd of 3?
   (iii) $\cdot 89$ nbd of 2?

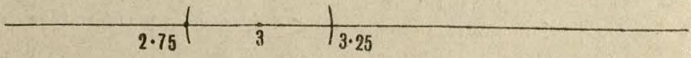
(i) $\cdot 01$ nbd of 4 $= (4 - \cdot 01, 4 + \cdot 01)$
$= (3 \cdot 99, 4 \cdot 01)$
$= \{x \mid 3 \cdot 99 < x < 4 \cdot 01\}$



Obviously, $x=3$ does not belong to this set

$\therefore\ 3 \notin (\cdot 01$ nbd of $4)$

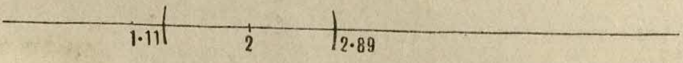
(ii) $\cdot 25$ nbd of 3 $= (3 - \cdot 25, 3 + \cdot 25)$
$= (2 \cdot 75, \quad 3 \cdot 25)$
$= \{x \mid 2 \cdot 75 < x < 3 \cdot 25\}$



Since $2 \cdot 75 < 3 < 3 \cdot 25$

$\therefore\ 3 \epsilon\ (\cdot 25$ nbd of $3)$

(iii) $\cdot 89$ nbd of 2 $= (2 - \cdot 89, 2 + \cdot 89)$
$= (1 \cdot 11, 2 \cdot 89)$
$= \{x \mid 1 \cdot 11 < x < 2 \cdot 89\}$



But $x=3$ does not satisfy the condition of an element of this set,
$\therefore\ 3 \notin (\cdot 89$ nbd of $2)$

2. If $y=10x$, and y lies in $\cdot 1$ nbd of 50, show that x lies in $\cdot 01$ nbd of 5.

$\because\ y \varepsilon\ \cdot 1$ nbd of 50

$\therefore \quad y \in (50 - \cdot 1,\ 50 + \cdot 1)$

$\therefore \quad 50 - \cdot 1 < y < 50 + \cdot 1$

$\therefore \quad 50 - \cdot 1 < 10x < 50 + \cdot 1 \quad (\because\ y = 10x)$

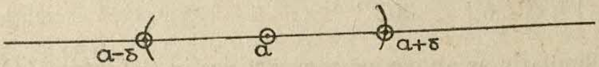
$\therefore \quad \frac{50}{10} - \frac{\cdot 1}{10} < x < \frac{50}{10} + \frac{\cdot 1}{10}$

$\therefore \quad 5 - \cdot 01 < x < 5 + \cdot 01$

$\therefore \quad x \in (5 - \cdot 01,\ 5 + \cdot 01)$

$\therefore$ x lies in ·01 nbd of 5.

*7·5. **Deleted neighbourhood :** Let 'a' be any point on the real line. We know that the δ-nbd of 'a' is the open interval $(a - \delta, a + \delta)$



that is, the set of all points between a−δ and a + δ (except the extreme points). Now suppose the point 'a' is taken out of this set. Then the open interval $(a-\delta,\ a+\delta)$ except the point a is called the Deleted δ-neighbourhood of a.

$$\begin{aligned}\text{Thus Deleted } \delta\text{-nbd of } a &= (a-\delta,\ a+\delta) \sim \{a\}\\ &= \{x \mid a-\delta < x < a+\delta \text{ and } x \neq a\}\\ &= \{x \mid\ |x-a| < \delta \text{ and } x-a \neq 0\}\\ &= \{x \mid 0 < |x-a| < \delta\}.\end{aligned}$$

*7·6. Let us now reconsider the two examples discussed in art. 7·1. In ex. 1, for values of x very close to x = 3, the values of f(x) are found to be very close to the number 6. We make the following observations from the table giving the values of f(x) corresponding to those of x :—

(1) When x lies in the interval (2·9999, 3·0001), f(x) lies in the interval (5·9998, 6·0002). Since a symmetric open interval about a point is a neighbourhood, we can say that when x lies in ·0001 nbd of 3, (x) lies in ·0002 nbd of 6. Or, in other words,

"Corresponding to ·0002 nbd of 6 there exists ·0001 nbd of 3 (except 3 itself) such that whenever x is in ·0001 nbd of 3 (except 3 itself), f(x) is necessarily in ·0002 nbd of 6." In short,

whenever $3 - \cdot 0001 < x < 3 + \cdot 0001$, we have

$$6 - \cdot 0002 < f(x) < 6 + \cdot 0002$$

that is, $0 < | x - 3 | < \cdot 0001$ implies

$$| f(x) - 6 | < \cdot 0002.$$

(2) Exactly as above, whenever x lies in the interval (2·999, 3·001), f(x) lies in (5·998, 6·002) and hence we have, "corresponding to ·002 nbd of 6 there exists ·001 nbd of 3 (except 3 itself) such that

$$0 < | x - 3 | < \cdot 001 \text{ implies}$$

$$| f(x) - 6 | < \cdot 002 \text{"}$$

(3) Similarly as above, whenever x lies in the interval (2·99, 3·01), f(x) lies in (5·98, 6·02), so that,

"corresponding to ·02 nbd of 6 there exists ·01 nbd of 3 (except 3 itself) such that $0 < | x - 3 | < \cdot 01$ implies

$$| f(x) - 6 | < \cdot 02 \text{"}$$

We can thus go on taking nbds of 6 and find that whatever nbd of 6 we take there exists some nbd of 3 (except 3 itseslf—that is, a deleted nbd of 3) such that when x lies in the deleted nbd of 3 thus obtained, f(x) will lie in the chosen nbd of 6.

In general, if $\epsilon > 0$ is any real number, corresponding to the ε nbd of 6 there exists a deleted δ — nbd of 3 ($\delta > 0$) such that $0 < | x - 3 | < \delta$ implies $| f(x) - 6 | < \epsilon$

This, then, is the rigorous mathematical way of writing

$$\lim_{x \to 3} f(x) = 6$$

which we had intuitively written before. The same phraseology can be formed in case of example 2 also, by taking some more readings before the value v = 24 (that is, for values of v less than 24) so as to get symmetrical intervals.

7·7. **Definition :** If f(x) be a areal valued function[1] of x then the real number l is said to be the limit of f(x) at x = a if and only if for ϵ nbd of 'l' there exists a corresponding δ-nbd of 'a' such that x ε δ-nbd any of a (but $x \neq a$) implies f(x) ϵ ϵ nbd of l.

Equivalently: For any real valued function f(x) of x, if for any $\epsilon > 0$, there exists $\delta > 0$ such that $0 < | x - a | < \delta$ implies $| f(x) - l | < \epsilon$ then l is said to be the limit of f(x) at x = a and we write $\lim\limits_{x \to a} f(x) = l$

Illustration : Given f(x) = 3x is a real valued function of x, test whether

$$\lim_{x \to 5} f(x) = 15$$

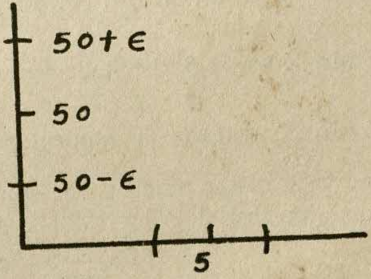
The problem is "given any $\epsilon > 0$, can we find a real no. $\delta > 0$ such that whenever x lies in the deleted δ-nbd of 5, f(x) will lie in the ϵ nbd of 15? For any $\epsilon > 0$, f(x) lies in ϵ nbd of 15 means

$$15 - \epsilon < f(x) < 15 + \epsilon$$

. e. $15 - \epsilon < 3x < 15 + \epsilon$

i. e. $\frac{15}{3} - \frac{\epsilon}{3} < x < \frac{15}{3} + \frac{\epsilon}{3}$

i. e. $5 - \frac{\epsilon}{3} < x < 5 + \frac{\epsilon}{3}$



So, if we take $\delta = \frac{\epsilon}{3}$ then $5 - \delta < x < 5 + \delta$.

We therefore have, $5 - \delta < x < 5 + \delta$ implies

$15 - \epsilon < f(x) < 15 + \epsilon$.

That is, x ϵ δ nbd of 5 implies f(x) ϵ ϵ— nbd of 15

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[1] A function of which each of the domain and the range is the set of real number is called a real valued function.

Also, Since $\epsilon > 0$, therefore $\frac{\epsilon}{3} > 0$, $\therefore \delta > 0$.

Thus, given any $\epsilon > 0$, we have a $\delta \left(= \frac{\epsilon}{3} \right) > 0$ such that $0 < | x - 5 | < \delta$ implies

$$| f(x) - 15 | < \epsilon$$

$$\therefore \lim_{x \to 5} f(x) = 15$$

Remarks : The $\epsilon - \delta$ technique is like a game being played between an unknown challenger say Mr. X and the student. Mr. X writes down any real number (which is ϵ), the student has to produce a corresponding number (which is δ) such that the condition for the limit is satisfied. In the above example, $\delta = \frac{\epsilon}{3}$ is the secret possessed by the student from which he finds out what number to produce to fit into the condition. Thus, if Mr. X writes 3, the student produces $\frac{3}{3}$ i. e. 1; if Mr. X writes 5, the student produces $\frac{5}{3}$; if Mr. X writes 6, the student produce $\frac{6}{3}$ i. e. 2; and each of the numbers the student produces satisfies every time the condition that $0 < | x - 5 | < \delta$ implies

$$| f(x) - 5 | < \epsilon.$$

7·8. **Theorems on limits :** The algebra of limits is given by the following theorems on limits of real valued functions. The proofs of these rems are outside the scope of this book. We shall, however, assume them theoto be true and discuss some of their applications.

1. $\lim_{x \to a} x = a$ and $\lim_{x \to a} k = k$, where k is a real constant
2. $\lim_{x \to a} [f_1 (x) + f_2 (x)] = \lim_{x \to a} f_1 (x) + \lim_{x \to a} f_2 (x)$.

3. $\lim\limits_{x \to a} [f_1(x) - f_2(x)] = \lim\limits_{x \to a} f_1(x) - \lim\limits_{x \to a} f_2(x)$.

4. $\lim\limits_{x \to a} \left[f_1(x)\, f_2(x) \right] = \left[\lim\limits_{x \to a} f_1(x) \right] \times \left[\lim\limits_{x \to a} f_2(x) \right]$

5. $\lim\limits_{x \to a} \dfrac{f_1(x)}{f_2(x)} = \dfrac{\lim\limits_{x \to a} f_1(x)}{\lim\limits_{x \to a} f_2(x)}$ provided $\lim\limits_{x \to a} f_2(x) \neq 0$

6. $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$, $x \neq 0$ and x is expressed in radians.

Throughout the discussion on limits hereon, the functions will always be real valued functions, unless otherwise specifically stated.

Examples :

1. Find (i) $\lim\limits_{x \to 5} (x + 21)$, (ii) $\lim\limits_{x \to b} (x^3)$

(iii) $\lim\limits_{x \to -2} (9x^2 - 13x + 2)$

(i) $$\begin{aligned} \lim_{x \to 5} (x + 21) &= \lim_{x \to 5} x + \lim_{x \to 5} 21 \quad \text{(Th. 2)} \\ &= 5 + 21 \quad \text{(Th. 1)} \\ &= 26 \end{aligned}$$

(ii) $$\lim_{x \to b} x^3 = \lim_{x \to b} (x \cdot x \cdot x)$$

$$= \left(\lim_{x \to b} x \right) \left(\lim_{x \to b} x \right) \left(\lim_{x \to b} x \right) \quad \text{(Th·4)}$$

$$= (b)(b)(b) \quad \text{(Th·1)}$$

$$= b^3$$

It can similarly be proved that $\lim\limits_{x \to b} x^n = b^n$

where n is any positive integer.

$$\text{(iii)}\ \lim_{x \to -2} (9x^2 - 13x + 2)$$

$$= \lim_{x \to -2} (-\ (_{2}x\ 6 \quad \lim_{x \to -2} (13x) + \lim_{x \to -2} (2) \quad (\text{Th.2 \& 3})$$

$$= \left(\lim_{x \to -2} 9\right) \left(\lim_{x \to -2} x^2\right)$$

$$- \left(\lim_{x \to -2} 13\right) \left(\lim_{x \to -2} x\right)$$

$$+ \lim_{x \to -2} (2) \quad (\text{Th. 4 \& 1})$$

$$= 9 \cdot (-2)^2 - 13 \cdot (-2) + 2 \quad (\text{Th} \cdot 1)$$

$$= 36 + 26 + 2$$

$$= 64$$

2. Find $\lim_{z \to c} \frac{z^3 - c^3}{z^2 - c^2}$, $z \neq c$.

$$\lim_{z \to c} \frac{z^3 - c^3}{z^2 - c^2} = \lim_{z \to ;} \frac{(z - c)(z^2 + zc + c^2)}{(z - c)(z + c)}$$

$$= \lim_{z \to c} \frac{z^2 + zc + c^2}{z + c} \quad \because z \neq c$$

$$= \frac{\lim_{z \to c} (z^2 + zc + c)}{\lim_{z \to c} z + c} \quad \text{Th} \cdot 5.$$

$$= \frac{c^2 + c \cdot c + c^2}{c + c}$$

$$= \frac{3c^2}{2c}$$

$$= \frac{3}{2} c$$

3. If $f(x) = \dfrac{x^2 - 4}{x - 2}$, $x \neq 2$,

and $f(2) = 5$,

find $\lim\limits_{x \to 2} f(x)$.

We have, $\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$

$$= \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2}$$

$$= \lim_{x \to 2} (x+2) \quad \because x \neq 2$$

$$= \lim_{x \to 2} x + \lim_{x \to 2} 2 \text{ (Th}\cdot\text{2)}$$

$$= 2 + 2 = 4.$$

Note that $\lim\limits_{x \to 2} f(x) = 4$, but $f(2) = 5$

Thus, in any example, $\lim\limits_{x \to a} f(x)$ and $f(a)$ may or may not be the same.

4. Find $\lim\limits_{x \to \alpha} \dfrac{\sin x - \sin \alpha}{x - \alpha}$

$$\lim_{x \to \alpha} \frac{\sin x - \sin \alpha}{x - \alpha} = \lim_{x \to \alpha} \frac{2 \sin \frac{x - \alpha}{2} \cos \frac{x + \alpha}{2}}{x - \alpha}$$

$$= \lim_{x \to \alpha} \frac{2 \,.\, \sin \frac{x - \alpha}{2}}{x - \alpha} \cdot \cos \frac{x + \alpha}{2}$$

$$= \lim_{x \to \alpha} \frac{2 \cdot \sin \frac{x - \alpha}{2}}{x - \alpha} \cdot \lim_{x \to \alpha} \cos \frac{x + \alpha}{2}$$

$$= \lim_{x \to \alpha} \frac{\sin \frac{x-\alpha}{2}}{\frac{x-\alpha}{2}} \cdot \lim_{x \to \alpha} \cos \frac{x+\alpha}{2}$$

Put $\frac{x-\alpha}{2} = y$ then, as $x \to \alpha$, $y \to 0$

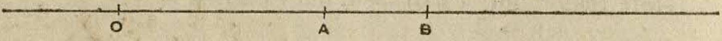
$$\therefore \lim_{x \to \alpha} \frac{\sin x - \sin \alpha}{x - \alpha} = \lim_{y \to 0} \frac{\sin y}{y} \cdot \lim_{x \to \alpha} \cos \frac{x+\alpha}{2}$$

$$= 1 \cdot \cos \frac{\alpha + \alpha}{2}.$$

$$= \cos \alpha.$$

7·9. The derivative :

In example 2 of art 7·1, the motion of a particle along a straight line OAB is discussed. The law of motion is given by $S = 3t^2$, where S is the distance of the particle from the point O at time t.



S is thus a function of t[1]. The value of S depends on the value of t. S is called the dependent variable and t the independent variable.

The velocity at any point A (which is the position of the particle at t = 4) is the rate of displacement at that point and, as we have seen, is given by

$$v_{\text{at A}} = \lim_{h \to 0} \frac{3(4+h)^2 - 3 \cdot 4^2}{h}$$

Suppose now that the specific law (connecting S and t) of the motion of a particle is not given, but we know that the displacement S is dependent on time t only. Then, since S is a function of t, we shall assume the law of motion to be

$$S = f(t)$$

f(t) standing for the unknown expression in t.

---

[1] It can be verified that $S = 3t^2$ defines a function. For, each value of corresponds to one and only one value of S. The set of values of t forms the domain and the set of values of S forms the range. The codomain is R itself

$$\therefore \quad S_{\text{at A}} = f(t) \text{ and } S_{\text{at B}} = f(t+h).$$

Since velocity at A is the limit of the difference of the displacements at A and B divided by h as $h \to 0$,

$$\therefore \quad V_{\text{at A}} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \quad \ldots\ldots \quad \text{(i)}$$

But velocity at A is the rate of displacement at A, that is, velocity at A is the rate of change of S with respect to t.

$\therefore$ rate of change of S at A with respect to t

$$= \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

The left hand side is denoted for convenience by $\frac{d}{dt} S$, or, equivalently by $\frac{d}{dt} f(t)$ ($\because S = f(t)$) and is called "derivative" of, or the "differential coefficient" of S with respect to t.

7·10. Definition : If $y = f(x)$ be any real valued function of x the rate of change of y with respect to x at the point (x, y) is then given by

$\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$. It is denoted by $\frac{d}{dx}$ f(x) or $\frac{d}{dx}$ y and is called the derivative of, or the differential coefficient of f(x) (or y) with respect to x.

$$\text{Thus, } \frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Remarks :

1. $\frac{d}{dx} f(x)$ does not mean it is a product of $\frac{d}{dx}$ with f(x). In fact 'dx' is not a real number and neither is 'd' so. The notation $\frac{d}{dx}$

is used as a symbol to denote the operation of differentiation and stands for the words "the differential coefficient of."

2. f is a function which maps the objects x into the images y. The domain is the set of values of x, and is R unless otherwise stated. The co-domain is also the set R. The range depends in each case on the actual law connecting x and y and is the set of values of y.

3. Frequently, instead of using h, we write $\triangle x$ or δx (read delta x") for h. In our discussion of derivatives hereafter, $\triangle x$ will mostly be used instead of h. The symbol $\triangle x$ (as also the symbol h) denotes an increment in x.

The symbols $\triangle y$ and $\triangle z$ will, therefore, under this convention, denote increments in y and z respectively.

4. Since a derivative is nothing but the rate of change of the dependent variable with respect to the independent variable, it is also called the "rate measurer".

5. The symbol $\frac{dy}{dx}$ is written in many other ways such as, y', $\frac{d}{dx} f(x)$, $f'(x)$, Dy etc.

6. It is to be noted that $\triangle x$ or δx or h is an *increment* in the independent variable x and is therefore a real number although it is very small. $\triangle x$ therefore obeys all the laws of the real number system and may be used for any operation defined in this system.

On the other hand dx is not a real number; it is merely a symbol expressed in the form $\frac{d}{dx}$, and stands for the operation of differentiation.

7. The process of finding the differential coefficient of a function from the definition is called finding the derivative 'from the first principles' or "ab initio".

7·11. Illustrative Examples :

(1) Find, from the first principles the derivative of the real valued function f(x) with respect to x given that $f(x) = x^2$ and hence find f(0), f(5), f(−3).

Here, $f(x) = x^2 \quad \therefore f(x + \triangle x) = (x + \triangle x)^2$

$$\therefore \frac{d}{dx} f(x) = \lim_{\triangle x \to 0} \frac{(x + \triangle x)^2 - x^2}{\triangle x} \quad \text{from definition.}$$

$$= \lim_{\triangle x \to 0} \frac{x^2 + 2x \cdot \triangle x + (\triangle x)^2 - x^2}{\triangle x}$$

$$= \lim_{\triangle x \to 0} \frac{\triangle x \{ 2x + \triangle x \}}{\triangle x}$$

$$= \lim_{\triangle x \to 0} (2x + \triangle x)$$

$$= \lim_{\triangle x \to 0} 2x + \lim_{\triangle x \to 0} \triangle x$$

$$= 2x$$

So, $f(0) = 2 \cdot 0 = 0; \quad f(5) = 2 \cdot 5 = 10,$

$f(-3) = 2 \cdot (-3) = -6$

(2) If $f(x) = 5x + 3$, $x \in R$, find $f'(2)$ and $f'\left(\frac{-1}{7}\right)$

Here, $f(x) = 5x + 3 \quad \therefore f(x + \triangle x) = 5(x + \triangle x) + 3$

$$\therefore \frac{d}{dx} f(x) = \lim_{\triangle x \to 0} \frac{\{5(x + \triangle x) + 3\} - \{5x + 3\}}{\triangle x}$$

$$= \lim_{\triangle x \to 0} \frac{5 \triangle x}{\triangle x}$$

$$= \lim_{\triangle x \to 0} 5$$

$$= 5.$$

$\therefore \quad f'(x) = 5$ hence $f'(2) = 5$ and $f'\left(-\frac{1}{7}\right) = 5$

Note that the derivative of a linear function is a constant.

(3) Find the derivative of $\frac{1}{x}$ from the first principles.

Here, $f(x) = \frac{1}{x} \quad \therefore \quad f(x + \triangle x) = \frac{1}{x + \triangle x}$

$$\therefore \frac{d}{dx} f(x) = \lim_{\triangle x \to 0} \frac{\frac{1}{x + \triangle x} - \frac{1}{x}}{\triangle x}$$

$$= \lim_{\triangle x \to 0} \frac{x - (x + \triangle x)}{x \cdot (x + \triangle x) \cdot \triangle x}$$

$$= \lim_{\triangle x \to 0} \frac{- \triangle x}{x \cdot (x + \triangle x) \cdot \triangle x}$$

$$= \lim_{\triangle x \to 0} \frac{(-1)}{x(x + \triangle x)}$$

$$= \frac{\lim\limits_{\triangle x \to 0} (-1)}{\lim\limits_{\triangle x \to 0} x \cdot \lim\limits_{\triangle x \to 0} (x + \triangle x)}$$

$$= \frac{-1}{x \cdot x}$$

$$= -\frac{1}{x^2}$$

7·12. Derivatives of some standard functions :

In the previous article we have seen that for any real valued function f(x) of x,

$$\frac{d}{dx} f(x) = \lim_{\triangle x \to 0} \frac{f(x + \triangle x) - f(x\}}{\triangle x}$$

With the help of this formula, we now proceed to find the derivative of some standard functions.

1. Derivative of x^n {n being a + ve integer)

here, $f(x) = x^n \quad \therefore f(x + \triangle x) = (x + \triangle x)^n$

$$\therefore \frac{d}{dx} x^n = \lim_{\triangle x \to 0} \frac{(x + \triangle x)^n - x^n}{\triangle x}$$

$$= \lim_{\triangle x \to 0} \frac{x^n \left(1 + \frac{\triangle x}{x} \right) - x^n}{\triangle x}$$

By Binomial theorem,

$$= \lim_{\triangle x \to 0} \frac{x^n \left[1 + n\frac{(\triangle x)}{x} + \frac{n(n-1)}{2!} \frac{(\triangle x)^2}{x^2} + \ldots + \frac{(\triangle x)^n}{x^n} \right] - x^n}{\triangle x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x \left[nx^{n-1} + \frac{n(n-1)\Delta x\, x^{n-2}}{2!} + \ldots + (\Delta x)^{n-1} \right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} nx^{n-1} + \lim_{\Delta x \to 0} \frac{n(n-1)}{2!} (\Delta x)\, x^{n-2} + \ldots$$

$$+ \lim_{\Delta x \to 0} (\Delta x)^{n-1}$$

$= nx^{n-1}$, Since all terms after the first vanish.

Thus, $\frac{d}{dx} x^{15} = 15\, x^{15-1} = 15\, x^{14}$,

$\frac{d}{dx} x^{24} = 24\, x^{24-1} = 24\, x^{23}$

2. Derivative of $x^{1/2}$

$f(x) = x^{1/2} \quad \therefore \quad f(x + \Delta x) = (x + \Delta x)^{1/2}$

$$\therefore \frac{d}{dx} x^{1/2} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^{1/2} - x^{1/2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^{1/2} - x^{1/2}}{\Delta x} \times \frac{(x + \Delta x)^{1/2} + x^{1/2}}{(x + \Delta x)^{1/2} + x^{1/2}}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x \{ (x + \Delta x)^{1/2} + x^{1/2} \}}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x \{ (x + \Delta x)^{1/2} + x^{1/2} \}}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\{ (x + \Delta x)^{1/2} + x^{1/2} \}}$$

$$= \frac{\lim_{\Delta x \to 0} 1}{\lim_{\Delta x \to 0} (x + \Delta x)^{1/2} + \lim_{\Delta x \to 0} x^{1/2}}$$

$$= \frac{1}{x^{1/2} + x^{1/2}}$$

$$= \tfrac{1}{2}\, x^{-1/2}$$

Note that $\frac{d}{dx} x^{1/2} = \frac{1}{2}\, x^{-1/2} = \frac{1}{2}\, x^{1/2\, -}$

So that the derivative of $x^{1/2}$ could have been obtained by the formula of (1) of this article. Though it will not be attempted in this book, it can be established that for any real number n, $\frac{d}{dx} x^n = nx^{n-1}$.

(3) Derivative of Sin x (x being in radians).

$$f(x) = \text{Sin } x. \quad \therefore f(x + \triangle x) = \text{Sin } (x + \triangle x)$$

$$\therefore \frac{d}{dx}\text{Sin } x = \lim_{\triangle x \to 0} \frac{\text{Sin}(x + \triangle x) - \text{Sin } x}{\triangle x}$$

$$= \lim_{\triangle x \to 0} \frac{\frac{1}{2} \text{ Cos } \frac{2x + \triangle x}{2} \cdot \text{Sin } \frac{\triangle x}{2}}{\triangle x}$$

$$= \lim_{\triangle x \to 0} \text{Cos}\left(x + \frac{\triangle x}{2} \right) \cdot \frac{\text{Sin } \frac{\triangle x}{2}}{\frac{\triangle x}{2}}$$

$$= \lim_{\triangle x \to 0} \text{Cos}\left(x + \frac{\triangle x}{2} \right) \cdot \lim_{\triangle x \to 0} \frac{\text{Sin } \frac{\triangle x}{2}}{\frac{\triangle x}{2}}$$

it has already been asserted that

$\lim_{\theta \to 0} \frac{\text{Sin } \theta}{\theta} = 1$. Now as $\triangle x \to 0$, $\frac{\triangle x}{2} \to 0$

$$\therefore \lim_{\triangle x \to 0} \frac{\text{Sin } \frac{\triangle x}{2}}{\frac{\triangle x}{2}} = \lim_{\frac{\triangle x}{2} \to 0} \frac{\text{Sin } \frac{\triangle x}{2}}{\frac{\triangle x}{2}} = 1$$

also, $\lim_{\triangle x \to 0} \text{Cos}\left(x + \frac{\triangle x}{2} \right) = \text{Cos } x$

Substituting in the above expression,

$$\frac{d}{dx} \text{Sin } x = \text{Cos } x$$

Note that x is to be in radians. In the derivatives of all the trigonometric unctions discussed hereafter, x is in radians unless otherwise stated.

4. Derivative of cos x

$$f(x) = \cos x \quad \therefore\ f(x + \triangle x) = \cos(x + \triangle x)$$

$$\frac{d}{dx}\cos x = \lim_{\triangle x \to 0} \frac{\cos(x + \triangle x) - \cos x}{\triangle x}$$

$$= \lim_{\triangle x \to 0} \frac{2 \sin \frac{2x + \triangle x}{2} . \sin\left(-\frac{\triangle x}{2}\right)}{\triangle x}$$

$$= \lim_{\triangle x \to 0} \sin\left(x + \frac{\triangle x}{2}\right) . \frac{\left\{-\sin \frac{\triangle x}{2}\right\}}{\frac{\triangle x}{2}}$$

$$= \lim_{\triangle x \to 0} -\sin\left(x + \frac{\triangle x}{2}\right) . \frac{\sin \frac{\triangle x}{2}}{\frac{\triangle x}{2}}$$

$$= \lim_{\triangle x \to 0} \left\{-\sin\left(x + \frac{\triangle x}{2}\right)\right\} . \lim_{\triangle x \to 0} \frac{\sin \frac{\triangle x}{2}}{\frac{\triangle x}{2}}$$

We have, $\lim_{\triangle x \to 0} \frac{\sin \frac{\triangle x}{2}}{\frac{\triangle x}{2}} = 1$

and $\lim_{\triangle x \to 0} \left\{-\sin\left(x + \frac{\triangle x}{2}\right)\right\} = -\sin x$

So that $\frac{d}{dx} \cos x = -\sin x$.

5. Derivative of a constant K.

Here $f(x) = K$.

$$\text{i·e}\quad f(x) = K\cdot x^{\circ} \quad \therefore\ f(x+\triangle x) = K\cdot(x+\triangle x)^{\circ} = K.$$

$$\begin{aligned}
\therefore\ \frac{d}{dx}K &= \lim_{\triangle x\to 0}\frac{K-K}{\triangle x}\\
&= \lim_{\triangle x\to 0}\frac{0}{\triangle x}\\
&\equiv \lim_{\triangle x\to 0} 0\\
&= 0
\end{aligned}$$

Thus, $\frac{d}{dx}\,5 = 0 = \frac{d}{dx}\,(215) = \frac{d}{dx}\left(\frac{1}{\sqrt[7]{39}}\right) = \frac{d}{dx}\,(-\sqrt{6})$

6. **Derivative of tan x.**

$$f(x) = \tan x \qquad \therefore f(x+\triangle x) = \tan(x+\triangle x)$$

$$\begin{aligned}
\therefore\ \frac{d}{dx}\tan x &= \lim_{\triangle x\to 0}\frac{\tan(x+\triangle x)-\tan x}{\triangle x}\\
&= \lim_{\triangle x\to 0}\frac{\dfrac{\sin(x+\triangle x)}{\cos(x+\triangle x)}-\dfrac{\sin x}{\cos x}}{\triangle x}\\
&= \lim_{\triangle x\to 0}\frac{\sin(x+\triangle x)\cos x-\sin x\cdot\cos(x+\triangle x)}{\cos(x+\triangle x)\cdot\cos x\cdot\triangle x}\\
&= \lim_{\triangle x\to 0}\frac{\sin(x+\triangle x-x)}{\cos(x+\triangle x)\cos x\cdot\triangle x}\\
&= \lim_{\triangle x\to 0}\frac{1}{\cos(x+\triangle x)\cdot\cos x}\cdot\frac{\operatorname{Sin}\triangle x}{\triangle x}\\
&= \frac{\lim\limits_{\triangle x\to 0} 1}{\lim\limits_{\triangle x\to 0}\cos(x+\triangle x)\cdot\lim\limits_{\triangle x\to 0}\cos x}\cdot\lim_{\triangle x\to 0}\left(\frac{\sin\triangle x}{\triangle x}\right)\\
&= \frac{1}{\cos x\cdot\cos x}\cdot 1\\
&= \frac{1}{\cos^2 x}\\
&= \sec^2 x.
\end{aligned}$$

7·13. Algebra of derivatives :

For the scope of our discussion, we shall assume the laws of derivatives for addition, subtraction, multiplication, and division of any two functions of x without proof.

Let $f_1(x)$ and $f_2(x)$ be any two real valued functions of x. Then the following are the rules giving the algebra of derivatives.

(i) Derivative of a sum.

$$\frac{d}{dx}\left\{f_1(x)+f_2(x)\right\}=\frac{d}{dx}f_1(x)+\frac{d}{dx}f_2(x)$$

i. e. the derivative of the sum of two functions is the same as the sum of the derivatives of the individual functions.

Thus $$\frac{d}{dx}\left\{\operatorname{Sin} x+x^9\right\}=\frac{d}{dx}\operatorname{Sin} x+\frac{d}{dx}x^9$$
$$=\operatorname{Cos} x+9x^8$$

(ii) Derivative of a difference. :

$$\frac{d}{dx}\left\{f_1(x)-f_2(x)\right\}=\frac{d}{dx}f_1(x)-\frac{d}{dx}f_2(x)$$

.e. the derivative of difference of two functions is equal to the difference of their derivatives. Thus,

$$\frac{d}{dx}\left\{x^{1/2}-\tan x\right\}=\frac{d}{dx}x^{1/2}-\frac{d}{dx}\tan x$$
$$=\frac{1}{2}x^{-1/2}-\operatorname{Sec}^2 x$$
$$=\frac{1}{2\sqrt{x}}-\operatorname{Sec}^2 x$$

Both the above rules can be extended to more than two functions.

(iii) Derivative of a product :

$$\frac{d}{dx}\left\{f_1(x)\cdot f_2(x)\right\}=f_1(x)\cdot\frac{d}{dx}f_2(x)+f_2(x)\cdot\frac{d}{dx}f_1(x)$$

i.e. The derivative of the product of two functions = (first function)

× (derivative of the second function) + (second function) × (derivative of the first function.)

$$\text{Thus,}\quad \frac{d}{dx}(x^3 \cdot \cos x) = x^3 \cdot \frac{d}{dx}\cos x + \cos x \cdot \frac{d}{dx}(x^3)$$
$$= x^3 \cdot (-\sin x) + \cos x \cdot 3x^2$$
$$= -x^3 \sin x + 3x^2 \cos x$$

This rule can be extended to the derivative of the product of more than two functions. Thus,

Let u, v, w be any three functions of x. Then,

$$\frac{d}{dx}\left(u\,v\,w\right) = \frac{d}{dx}\left\{(u\,v)\,w\right\}$$

$$\text{(using product rule)} \quad = (u\,v)\frac{d}{dx}w + w\frac{d}{dx}(u\,v)$$
$$= u\,v\frac{dw}{dx} + w\left\{u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$
$$= v\,w\frac{du}{dx} + u\,w\frac{dv}{dx} + u\,v\frac{dw}{dx}$$

Similarly, for any n functions $u_1, u_2 \ldots\ldots\ldots u_n$ of x,

$$\frac{d}{dx}(u_1 \cdot u_2 \ldots\ldots\ldots u_n) = u_2\,u_3 \ldots u_n \frac{du_1}{dx}$$
$$+ u_1\,u_3 \ldots u_n \frac{du_2}{dx} + \ldots$$
$$+ u_1\,u_2 \ldots u_{n-1}\frac{du_n}{dx}$$

$$\text{Thus}\quad \frac{d}{dx}\left(x^{3/2}\sin x\right) = \frac{d}{dx}\left(x \cdot x^{1/2} \cdot \sin x\right)$$
$$= x^{\frac{1}{2}}\sin x.\frac{d}{dx}(x) + x\sin x\frac{d}{dx}(x^{1/2})$$

$$+ x \cdot x^{1/2} \frac{d}{dx} (\text{Sin } x)$$

$$= x^{1/2} \text{ Sin} x \cdot 1 + x \text{ Sin } x \cdot \frac{1}{2} \cdot x^{-1/2} + x^{3/2} \text{ Cos } x.$$

(iv) Derivative of the product of a constant and a function of x.

Let k be any constant and f(x) be a function of x.

$$\text{Then, } \frac{d}{dx} \{ k f(x) \} = \lim_{\Delta x \to 0} \frac{k f(x + \Delta x) - k f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} k \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} k . \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} \right]$$

$$= k \cdot \frac{d}{dx} f(x)$$

$$\text{Thus, } \frac{d}{dx} (5 x^7) = 5 \cdot \frac{d}{dx} x^7$$

$$= 5 \cdot 7 x^6$$

$$= 35 x^6$$

$$\frac{d}{} (- \text{ Sin } x) = \frac{d}{dx} \left[(-1) \cdot \text{ Sin } x \right]$$

$$= (-1) \frac{d}{dx} \text{ Sin } x$$

$$= - \text{ Cos } x$$

(v) Derivative of a quotient ·

$$\frac{d}{dx} \left\{ \frac{f_1(x)}{f_2(x)} \right\} = \frac{f_2(x) \frac{d}{dx} f_1(x) - f_1(x) \frac{d}{dx} f_2(x)}{\{ f_2(x) \}^2}$$

Thus, the derivative of a quotient

$$= \frac{(\text{Denominator}) \times \frac{d}{dx}\text{numerator}) - (\text{numerator}) \times \frac{d}{dx}(\text{denominator})}{(\text{denominator})^2}$$

$$\text{Thus,}\quad \frac{d}{dx}\left(\frac{\tan x}{x^4}\right) = \frac{x^4 \frac{d}{dx}\tan x - \tan \frac{d}{dx} x^4}{(x^4)^2}$$

$$= \frac{x^4 \sec^2 x - \tan x \, 4x^3}{x^8}$$

$$= \frac{x \sec^2 x - 4 \tan x}{x^5}$$

7·14. Some more standard functions

With the help of the quotient rule, we shall now find the derivative of some more standard functions and extend the rule for the derivative of x^n to all integral values of n.

1 Derivative of cot x

$$\frac{d}{dx}\cot x = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$

$$= \frac{\sin x \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(\sin x)}{(\sin x)^2}$$

$$= \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x}$$

$$= -\operatorname{cosec}^2 x$$

2 Derivative of sec x :

$$\frac{d}{dx}\sec x = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$= \frac{\cos x \dfrac{d}{dx}(1) - 1 \dfrac{d}{dx}(\cos x)}{(\cos x)^2}$$

$$= \frac{(\cos x) \times 0 - 1 \times (-\sin x)}{\cos^2 x}$$

$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$

$$= \sec x \cdot \tan x$$

3. Derivative of cosesc x

It is left for the students to prove that

$$\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x.$$

4. Derivative of x^n where n is zero or a positive or negative integer.

If n is zero, then the derivative of n is the same as the derivative of a constant and hence is equal to zero.

$$\text{For,} \quad \frac{d}{dx} x^0 = \frac{d}{dx}(1) = 0$$

If n is a +ve integer, then as has already been proved,

$$\frac{d}{dx} x^n = nx^{n-1}$$

We shall consider the case when n is a −ve integer.

Let n = −m where m is a +ve integer.

$$\therefore \frac{d}{dx} x^n = \frac{d}{dx} x^{-m}$$

$$= \frac{d}{dx}\left(\frac{1}{x^m}\right)$$

$$= \frac{x^m \dfrac{d}{dx}(1) - (1)\dfrac{d}{dx}(x^m)}{(x^m)^2} \qquad \text{By quotient rule}$$

$$= \frac{x^m \times 0 - 1 \cdot m x^{m-1}}{x^{2m}}$$

$$= - m\, x^{-m-1}$$

$$= n x^{n-1} \quad \because n = -m$$

Thus whether n is zero or +ve or −ve integer,

$$\frac{d}{dx} x^n = n x^{n-1}$$

The quotient rule can be applied similarly to prove that

$$\frac{d}{dx} x^{-1]3} = - \frac{1}{2} x^{-3/2}$$

7·15. Illustrative Examples :

Find the derivatives of :

(i) $3x^2 - 5x + 10$ (ii) $\left(\sqrt{x} - x \right)^2$

(iii) $(x^2 - 5)\sqrt{x}$ (iv) $\frac{5x^2 + 7x + 6}{3x^2 + 2x + 9}$

(v) $\frac{6x^5}{\text{Sin}\, x \cdot \text{Cos}\, x}$

(i) $$\frac{d}{dx}(3x^2 - 5x + 10) = \frac{d}{dx}(3x^2) - \frac{d}{dx}(5x) + \frac{d}{dx}(10)$$

$$= \frac{3d}{dx}(x^2) - \frac{5d}{dx}(x) + 0$$

$$= 3 \cdot 2x - 5 \cdot 1$$

$$= 6x - 5$$

(ii) $$\frac{d}{dx}\left(\sqrt{x} - \frac{1}{x}\right)^2 = \frac{d}{dx}\left(x - 2\sqrt{x}\,\frac{1}{x} + \frac{}{x^2}\right)$$

$$= \frac{d}{dx}(x - 2x^{-1/2} + x^{-2})$$

$$= 1 - 2\left(\frac{-1}{2}\right) x^{-3/2} + (-2)x^{-3}$$

$$= 1 + x^{-3/2} - 2\,x^{-3}$$

(iii) $$\frac{d}{dx}(x^2 - 5)\sqrt{x} = \frac{d}{dx}\left(x^2\sqrt{x} - 5\sqrt{x}\right)$$

$$= \frac{d}{dx}(x^{5/2} - 5\,x^{1/2})$$

$$= \frac{5}{2}\;x^{3/2} - 5 \cdot \frac{1}{2}\,x^{-1/2}$$

$$= \frac{5}{2}\,[x^{3/2} - x^{-1/2}]$$

(iv) $$\frac{d}{dx}\left(\frac{5x^2 + 7x + 6}{3x^2 + 2x + 9}\right)$$

$$= \frac{(3x^2 + 2x + 9)\frac{d}{dx}(5x^2 + 7x + 6) - (5x^2 + 7x + 6)\frac{d}{dx}(3x^2 + 2x + 9)}{(3x^2 + 2x + 9)^2}$$

$$= \frac{(3x^2 + 2x + 9)\,(10x + 7) - (5x^2 + 7x + 6)\,(6x + 2)}{(3x^2 + 2x + 9)^2}$$

$$= \frac{-11x^2 + 54x + 51}{(3x^2 + 2x + 9)^2}$$

(v) $$\frac{d}{dx}\,(6x^5/\sin\,x\,\cos\,x) = \frac{d}{dx}\,(\,6x^5\,\mathrm{cosec}\,x\,\sec\,x\,)$$

$$= 6\,\frac{d}{dx}\,(\,x^5\,\mathrm{cosec}\,x\,\sec x\,)$$

$$= 6\left[\mathrm{cosec}\,x\,\sec\,x\,\frac{d}{dx}\,x^5 + x^5\sec x\,\frac{d}{dx}\,\mathrm{cosec}\,x + x^5\mathrm{cosec}\,x\,\frac{d}{dx}\,\sec\,x\right]$$

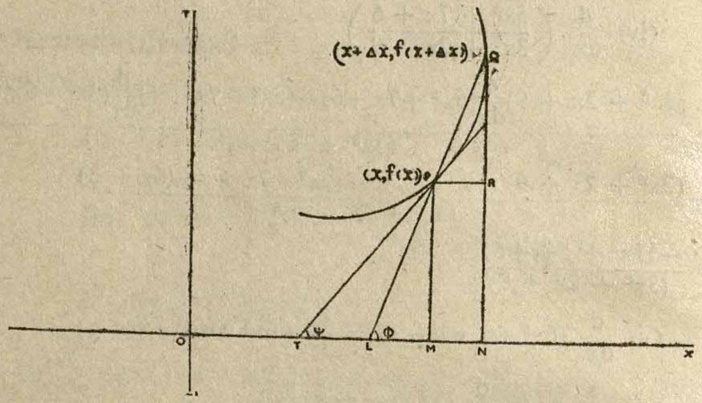
$$= 6\left[\mathrm{cosec}\,x\,\sec x \cdot 5x^4 + x^5\,\sec x\,(-\,\mathrm{cosec}\,x\,\cot x) + x^5\,\mathrm{cosec}\,x.\;\sec x.\;\tan x\right]$$

$$= 6\,x^4\,\sec x\,\mathrm{cosec}\,x\;[\,5 - x\,\cot x + x\,\tan x]$$

7·16. Geometrical meaning of derivative :

We have seen before that any expression of the form $y = f(x)$ where f is a real valued function, can be interpreted geometrically to represent a curve. We now proceed to show that the derivative s of y with respect to x provides a method for determining the slope of the curve $y = f(x)$ at any point (x, y).

Let $P(x, f(x))$ and $Q\,(x + \triangle x, f(x + \triangle x))$ be two points close to each other on the curve $y = f(x)$ drawn in the rectangular cartesian reference frame x′ox and y′oy. Let the chord PQ and the tangent at P make angles ϕ and ψ respectively with the axis of x. Also let PM and QN be perpendiculars drawn to the x-axis from P and Q respectively and PR be perpendicular to QN.



Then, $OM = x,\ ON = x + \triangle x,\ PM = f(x)$

$QN = f(x + \triangle x)$

So,
$$\begin{aligned} PR &= MN = ON - OM \\ &= x + \triangle x - x \\ &= \triangle x \end{aligned}$$

and $QR = QN - RN = QN - PM = f(x + \triangle x) - f(x)$

In $\triangle$ PQR, $\angle QPR = \phi$

$$\therefore \tan\phi = \frac{QR}{PR} = \frac{f(x + \triangle x) - f(x)}{\triangle x} \qquad \ldots\ldots()$$

Now let Q→P. Then the direction of the chord PQ→ the direction of the tangent at P and $\phi \to \psi$. Also, as Q → P, the co-ordinates of Q tend to become equal to the coordinates of P so that $x + \Delta x \to x$ and hence $\Delta x \to o$

$$\therefore \text{ from (i), } \lim_{\phi \to \psi} \tan \phi = \lim_{\Delta x \to o} \frac{f(x + \Delta x) - (x)}{\Delta x}$$

$$\therefore \tan \psi = \frac{d}{dx} f(x)$$

Thus, $\frac{d}{dx} f(x)$, i. e $\frac{dy}{dx}$ represents the trigonometric tangent of the angle made by the geometric tangent at the point (x, y) with the x axis. This quantity (that is, the trigonometric tangent of the angle made by a line with the x axis) is called the gradient or the slope of a line. Therefore, the derivative at any point represents the slope of the tangent to the curve at that point.

This provides a very useful and accurate method of finding the slope of the tangent at any point and therefore the nature of the curve at that point.

Remarks :

1. If the tangent at a point of a curve is parallel to the x - axis then $\psi = 0 \ \therefore \tan \psi = 0$.

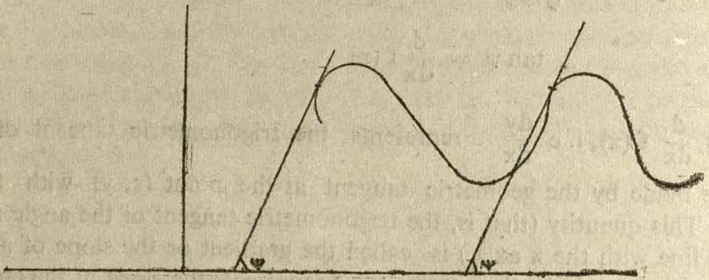
$$\therefore \frac{dy}{dx} = 0$$

Thus if at any point of the curve, $\frac{dy}{dx} = o$ then the tangent at that point is parallel to the axis of x and vice versa.

2. Just as the derivative of y with respect to x is the trigonometric tangent of the angle made by the tangent at the point with the x axis, similarly the trigonometric tangent of the angle made by the tangent at the point with the y axis is the derivative of x with respect to y. In short, if θ is the angle made by the tangent at (x, y) with the y axis,

then, $\tan\theta = \frac{dx}{dy}$.

$\therefore \frac{dx}{dy} = 0$ is the condition for the tangent at the point (x, y) to be parallel to the y-axis.



3. If $\frac{dy}{dx}$ has the same value at two points (or even more) of a curve then the tangents at these two points are parallel to each other.

Illustrations :

(1) Find the slope of the tangent at the point $\left(\frac{1}{2}, \frac{1}{4}\right)$ of the curve $y = x^2$.

$$\because y = x^2, \therefore \frac{dy}{dx} = \frac{d}{dx}x^2 = 2x$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at }(1/2,\ 1/4)} = (2x)_{\text{at } x = 1/2} = 2\cdot\frac{1}{2} = 1$$

The slope of the tangent at $\left(\frac{1}{2}, \frac{1}{4}\right)$ is therefore 1.

This means $\tan\psi = 1$ i, e. $\psi = \frac{\pi}{4}$ at $\left(\frac{1}{2}, \frac{1}{4}\right)$. So, the tangent at $\left(\frac{1}{2}, \frac{1}{4}\right)$ is inclined at an angle $\frac{\pi}{4}$ to the x - axis.

(2) Find the points on the curve $y = x^2 + 3x$ where the slopes are given by (i) 11, (ii) 4, (iii) 3, (iv) 0

Since $y = x^2 + 3x \therefore \frac{dy}{dx} = 2x + 3$

(i) $\frac{dy}{dx} = 11 \quad \therefore \quad 2x + 3 = 11$

$$\therefore \quad x = \frac{11-3}{2}$$

$$= 4$$

When $x = 4$, the equation $y = x^2 + 3x$

yields $y = 4^2 + 3 \cdot 4$

$= 28$

$\therefore$ The point is (4, 28)

(ii) $\frac{dy}{dx} = 4 \quad \therefore \quad 2x + 3 = 4$

$$\therefore \quad x = \frac{1}{2}$$

When $x = \frac{1}{2}$ the equation $y = x^2 + 3x$

yields $y = \left(\frac{1}{2}\right)^2 + 3 \cdot \frac{1}{2}$

$= \frac{7}{4}$.

$\therefore$ The point is $\left(\frac{1}{2}, \frac{7}{4}\right)$

(iii) $\frac{dy}{dx} = 3 \quad \therefore \quad 2x + 3 = 3 \quad \therefore \quad x = 0$

$\therefore$ by the same method as used in the above cases, $y = 0$

$\therefore$ The point is (0, 0); that is, the origin.

v) $\frac{dy}{dx} = 0 \quad \therefore \quad 2x + 3 = 0$

$$\therefore \quad x = -\frac{3}{2}$$

$$\therefore \quad y = \left(-\frac{3}{2}\right)^2 + 3\left(-\frac{3}{2}\right)$$

$$= -\frac{9}{4}.$$

$\therefore$ The point is $\left(-\frac{3}{2}, -\frac{9}{4}\right)$. Thus at the point $\left(-\frac{3}{2}, -\frac{9}{4}\right)$ the tangent to the curve $y = x^2 + 3x$ is parallel to the x-axis.

(3) The equation to a curve is $y = x^3 - 7x^2 + 5x - 2$.

Find the point of the curve where the tangent is parallel to the tangent at $(1, -3)$.

The curve is $y = x^3 - 7x^2 + 5x - 2$

$\therefore$ Slope at any point is given by

$$\frac{dy}{dx} = 3x^2 - 14x + 5$$

$$\text{at } (1, -3), \ \frac{dy}{dx} = 3 - 14 + 5$$

$$= -6$$

at the point where the tangent is parallel to $(1, -3)$, the slope must be the same. $\therefore$ We have to find the point where the slope is -6.

i. e. $\frac{dy}{dx} = -6$

i. e. $3x^2 - 14x + 5 = -6$

i. e. $3x^2 - 14x + 11 = 0$

i. e. $(x - 1)(3x - 11) = 0$

i. e. $x = 1$ or $\frac{11}{3}$

If $x = 1$ then $y = x^3 - 7x^2 + 5x - 2$ yields $y = -3$

i. e. point is $(1, -3)$ which is the given point where the slope is -6.

If $x = \frac{11}{3}$ then from the equation to the curve, $y = -\frac{769}{27}$

$\therefore$ at the point $\left(\frac{11}{3}, -\frac{769}{27}\right)$ the slope is -6. Therefore, at this point the tangent is parallel to the tangent at the point $(1, -3)$

7·17. Velocity and Acceleration :

As has already been observed in art 1·70 (Remark 4), the derivative $\frac{dy}{dx}$ is the rate of change of y with respect to x. If the independent variable is the time denoted by t and the dependent variable is s, the distance of a moving particle from a fixed point, then $\frac{ds}{dt}$ represents the rate of change of position with respect to t. In other words, $\frac{ds}{dt}$ is the rate of displacement, or as we generally call it, the velocity.

$$\text{Thus, } v = \frac{ds}{dt} \quad \ldots \quad \ldots \quad \text{(i)}$$

Similarly, the acceleration which is the rate of change of velocity is given

$$\text{by, } f = \frac{dv}{dt} \quad \ldots \quad \ldots \quad \text{(ii)}$$

The expressions (i) and (ii) are very useful and we shall shortly discuss some of the many types of problems in which they are used. The expression (ii) can be written in another way too. Since $v = \frac{ds}{d}$

$$\therefore f = \frac{d}{dt}\left(\frac{ds}{dt}\right).$$

The right hand side of this expression is the derivative of the derivative of s. This is denoted often by $\frac{d^2s}{dt^2}$ and is called the second derivative of s with respect to t. Thus, if a particle moves along a curve given by the equation $S = 3t^3 - 2t + 11$ then, to find the acceleration at time $t = 2$, we proceed as under : $S = 3t^3 - 2t + 11$

$$\therefore \frac{ds}{dt} = \frac{d}{dt}(3t^3 - 2t + 11)$$

$$= 9t^2 - 2$$

$$\therefore f = \frac{d^2s}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right)$$

$$= \frac{d}{dt}(9t^2 - 2)$$

$$= 18t .$$

$\therefore$ at $t = 2$, the acceleration $= 18 \times 2$

$= 36$ units.

In general, if y is a function of x [say $y = f(x)$] then $\frac{d^2y}{dx^2}$ denotes the second derivative of y with respect to x and is obtained by differentiating the expression for $\frac{dy}{dx}$ once again with respect to x.

Thus, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$.

7·18. The chain rule of derivation :

Sometimes the dependent variable y is not given directly as a function of x. Instead, y is given as a function of some other variable, say z while z in its turn is given as a function of the variable x. In such cases, the derivative of y with respect to x can be found by the formula derived as under and the process is called the chain rule of differentiation.

We shall assume that z and x increase or decrease simultaneously and so do y and z.

Then, since y is a function of z and z is a function of x, therefore to any increament $\triangle x$ of x will correspond on increament $\triangle z$ of z, and thereby an increament $\triangle y$ of y. Also, since $\triangle x$, $\triangle y$, $\triangle z$ are nothing but real numbers, we have,

$$\frac{\triangle y}{\triangle x} = \frac{\triangle y}{\triangle z} \times \frac{\triangle z}{\triangle x}.$$

$$\therefore \lim_{\triangle x \to 0} \left(\frac{\triangle y}{\triangle x}\right) = \lim_{\triangle x \to 0} \left(\frac{\triangle y}{\triangle z} \times \frac{\triangle z}{\triangle x}\right)$$

$$= \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle z} \times \lim_{\triangle x \to 0} \frac{\triangle z}{\triangle x}$$

But as $\triangle x \to 0$ we shall have $\triangle z \to 0$,

So that $\lim\limits_{\triangle x \to 0} \frac{\triangle y}{\triangle z} = \lim\limits_{\triangle z \to 0} \frac{\triangle y}{\triangle z}$.

and hence $\lim\limits_{\triangle x \to 0} \left(\frac{\triangle y}{\triangle x}\right) = \lim\limits_{\triangle z \to 0} \frac{\triangle y}{\triangle z} \times \lim\limits_{\triangle x \to 0} \frac{\triangle z}{\triangle x}$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

Illustrations :

1. Differentiate $\frac{1}{3 - 2x^2}$ with respect to x.

Let $y = \frac{1}{3 - 2x^2}$. We require $\frac{dy}{dx}$

Put $3 - 2x^2 = z$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(3-2x^2)^{-1} = \frac{d}{dx}(z^{-1})$$

$$= \frac{d}{dz}(z^{-1}) \times \frac{dz}{dx} = \frac{d}{dz}(z^{-1}) \times \frac{d}{dx}(3-2x^2)$$

$$= (-1)(z^{-2})(-4x)$$

$$= \frac{4x}{(3-2x^2)^2}$$

2. Find the gradient of the curve $x^{2/3} + y^{2/3} = 5$ at the point (8,1).
Differentiate $x^{2/3} + y^{2/3} = 5$ throughout with respect to x.

$$\therefore \quad \frac{d}{dx} x^{2/3} + \frac{d}{dx} y^{2/3} = \frac{d}{dx} 5 .$$

$$\therefore \quad \frac{2}{3} x^{(2/3-1)} + \frac{d}{dy} y^{2/3} \times \frac{dy}{dx} = 0$$

$$\therefore \quad \frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} \frac{d}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}$$

$$= -\left(\frac{y}{x}\right)^{1/3}$$

at (8,1), $\frac{dy}{dx} = -\left(\frac{1}{8}\right)^{1/3} = -\frac{1}{2}$

Hence the required gradient is $\frac{-1}{2}$.

EXERCISE

1. Express the following intervals as sets in tabular form.

(i) $[3, 5]$ (ii) $(-6, 0]$ (iii) $(-5, -1)$ (iv) $[-2, 4)$
(v) (a, y) (vi) $(c, n]$ (vii) $[-k, k]$

2. Compute the following :

(i) $(-1, 2) \cup [2, 5)$
(ii) $[0, 5] \cup (5, 8]$
(iii) $[6, 7) \cup [7, 9]$
(iv) $(-2, -1) \cup [-1, 0]$
(v) $(-5, -3) \cup (-3, -1)$
(vi) $(2, 9) \cup (5, 11)$
(vii) $(4, 13) \cup (-1, 7]$
(viii) $(-8, 2) \cup [0, 3]$
(ix) $[-1, 3] \cup [1, 3)$
(x) $(-2, 5) \cup [0, 2]$
(xi) $(2, 9) \cap (7, 11)$
(xii) $[4, 6] \cap [5, 8]$

(xiii) $(0, 3) \cap [3, 4)$ (xiv) $(-1, 2] \cap (2, 7]$

(xv) $[-5, 1) \cap (-2, 0)$

3. State whether each of the following is true or false:
   (i) $(5, 7) \subset (2, 7)$
   (ii) $[-1, 0] \subset (-2, 1)$
   (iii) $(-3, 3] \subseteq (-4, 3)$
   (iv) $(a, b] \cup (b, c] \cup (c, d) = (a, d)$
   (v) $[a, b) \cup [b, c) \cup [c, d] = [a, d]$
   (vi) $(a, b) \cap (b, c) = (a, c)$
   (vii) $(a, b) \cap [b, c) = \phi$.

4. Find whether the following numbers belong to the $\frac{1}{5}$ nbd of 9:

$8 \cdot 9, 9 \cdot 05, 9 \cdot 3, 9 \cdot 009, 8 \cdot 79, 8 \cdot 08, 8 \cdot 88.$

5. Given three neighbourhoods:

$A = \frac{1}{3}$ nbd. of 5

$B = \cdot 02$ nbd. of 7

$C = \frac{1}{2}$ nbd. of 49.

Find to which of these neighbourhoods the following real numbers belong:

$$48 \cdot 8,\ 49\frac{1}{3},\ 5 \cdot 2,\ 4 \cdot 9,\ 7\frac{1}{10},\ 5\frac{1}{4},\ \frac{243}{5},\ 6 \cdot 99,\ 7 \cdot 005.$$

6. Evaluate the following limits :

(i) $\lim\limits_{x \to 1} \frac{x^2 - 1}{x - 1}$, (ii) $\lim\limits_{x \to 2} \frac{x^4 - 3x^2 - 4}{x^2 + x - 6}$

(iii) $\lim\limits_{x \to 0} \frac{x}{2 - \sqrt{4 - x}}$ (iv) $\lim\limits_{x \to 0} \frac{x^2 + 2}{x^2 + 1}$

(v) $\lim\limits_{x \to 1} \frac{(1 + x)^m - 1}{x}$ (vi) $\lim\limits_{x \to 2} \frac{\sqrt{x^2 - 3} - 1}{x - 4}$.

(vii) $\lim\limits_{x \to \sqrt{2}} \dfrac{5x^4 - 6x^2 + 3}{x^2 + 1}$ (viii) $\lim\limits_{x \to -3} \dfrac{x^3 + 3x^2}{x^2 - 9}$

(ix) $\lim\limits_{x \to -\frac{1}{2}} \dfrac{2x + 1}{2x^3 + 5x^2 - 4x - 3}$

(x) $\lim\limits_{x \to a} \dfrac{\sqrt{3a - x} - \sqrt{2x}}{x^2 - a^2}$

7. Find the value of each of the following limits :

(i) $\lim\limits_{x \to \propto} \dfrac{\cos x - \cos \propto}{x - \propto}$ (ii) $\lim\limits_{x \to 0} \dfrac{2(1 - \cos x)}{x^2}$

(iii) $\lim\limits_{x \to 0} \dfrac{\sin mx}{\sin nx}$ (iv) $\lim\limits_{x \to \frac{\pi}{4}} \left[\dfrac{\sqrt{2} - \cos x + \sin x}{(4x + \pi)^2} \right]$

(v) $\lim\limits_{x \to \pi} \left(\dfrac{1 + \cos^3 x}{\tan^2 x} \right)$ (vi) $\lim\limits_{x \to 0} \left(\dfrac{1 - \cos x}{x \sin x} \right)$

(vii) $\lim\limits_{x \to a} \dfrac{\tan x - \tan a}{x - a}$

8. Differentiate the following functions with respect to x from the first principles. (a) $2x + 3$ (b) $x^2 - 4x + 5$ (c) $\sin 5x$ (d) $\tan 3x$ (e) $\sec x$.

9. Differentiate with respect to x

(i) x^{-13} , x^{25}, $\dfrac{1}{x^9}$

(ii) $\dfrac{15}{x^{23}}$, $27x^7$, $\dfrac{-5}{3x^2}$

(iii) $3 \cos x + 9x^4$

(iv) $5 \tan x - \dfrac{3}{x^5} - \sin x + 4$

(v) $kx^n - b^m + 3$

(vi) $(cx)^m - 8$,

(vii) $x^{10} \sin x \cos x$

(viii) $\sqrt[3]{15x^5}\tan x$

(ix) $5x^{17}\sec x + 20$

(x) $\dfrac{3\sin x}{x}$

(xi) $\dfrac{\cos x+2}{\sin x}$

(xii) $\dfrac{3-\tan x}{ax+b}$.

10. Find the derivative of tan x by the quotient rule.

11. Use the chain rule to differentiate the following functions with respect to x.

(i) $\left(x+\dfrac{1}{x}\right)^4$, (ii) $\sqrt{3x-5}$, (iii) $\dfrac{1}{1-9x}$

(iv) $\dfrac{7}{4x+15}$, (v) $\dfrac{1}{(x-1)^3}$, (vi) $\sin x^3 - \dfrac{2}{3x^2+1} - 7$

(vii) $10\tan^3 x + x^2\sin x$, (viii) $\dfrac{x^3}{(x^2+5)^{1/2}}$

(ix) $\sin(ax^3+bx^2+d)^4$, (x) $\dfrac{\sec^2 x}{5x^3+9}$

(xi) $(px^2+2qx+t)^{7/2}$, (xii) $\dfrac{1-\sin 3x}{\cos 5x}$.

(xiii) $\sqrt{\tan 7x}$, (xiv) $35\sin x^m - \sqrt{(a+bx)^p}$

(xv) $\dfrac{5\cos x}{(x^n+a)^q}$, (xvi) $\cos^4 x\sin^2 x$,

(xvii) $\dfrac{x^2}{(1-x^4)}$, (xviii) $\dfrac{2x+5}{x-9}$,

(xix) $(3x-1)\sqrt{x^3+2}$, (xx) $10(x+a)^6\cos(bx+d)^3$

12. Prove that $\dfrac{d}{dx}x^{-3} = (-3)x^{-4}$ with the help of the result $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -x^{-2}$.

13. Find the slopes of the following curves at the given points :

(i) $x^{1/2} + y^{1/2} = 5$ at (4, 9)

(ii) $2x^2 + 3y^2 = 11$ at (2, 1)

(iii) $\frac{1}{x} + \frac{9}{y} = 4$ at (1, 3)

(iv) $x^3 - y^3 = 9a^3$ at (a, 2a)

(v) $xy = 1$ at (6, – 2)

14. The surface and volume of a sphere of radius r are given by $S = 4\pi r^2$, and $V = \frac{4}{3}\pi r^3$ respectively. If the rate of increase of radius is 0·15 cm/sec, find at what rate the surface increases when r = 5 cms.

15. A wire of unknown resistance is joined in series with an ammeter and a battery of negligible resistance and in parallel with a voltmeter. The ammeter shows a reading of 0·2 amperes when the voltmeter shows 1·20 volts. At what rate should the current be changed to get a change of potential difference at the rate of ·3 volts/sec?

$$\left[\text{Use Ohm's law : } \frac{\text{potential difference}}{\text{current}} = \text{Resistance}\right]$$

16. A gas at constant temperature is contained in a vessel fitted with a movable air-tight piston. The volume of the gas is 25 cm^3 when the pressure is 4 cms. The piston is moved up so that the volume increases at the rate of 0·2 cm^3/sec. Find the rate of change of pressure when the volume is 20 cm^3. Explain the sign of your answer.

[Hint : Use Boyle's law : PV = constant]

17. If $g(x) = 3x^5 - 6x^4 + 4x + 2$, find $g^1(-2)$ and $g^1(3)$

18. The volume of a water pitcher is given by $2h^3 - 10h$ where h is the height occupied by water. What will be the rate of rise

of the water level when the depth of water in it is 5 cms if water is being poured into it at the rate of 28 cm^3/sec?

19. The object distance u, the image distance v and the focal length f of a lens are connected by the formula $\frac{1}{v} - \frac{1}{u} =$. An object, placed at a distance 4 cms from the lens on its axis moves away with a velocity 8 cms/sec. along the axis. Find the velocity of the image if it is formed at a distance 3 cms from the lens. [Hint : f is constant]

20. A car starting with a speed of 20 m. p. h. drives in city area for an hour, then turns over to a highway where the distance it covers from the starting point is directly proportional to the cube of the time in hours. Find its velocity and acceleration after half an hour.

21. Find the velocity and acceleration of a particle moving along a straight line after (i) 2 secs, (ii) 5 secs; in each of the following cases : (a) $S = 5t^2$

    (b) $3S = 10 - 6t + 3t^2$
    (c) $S = 2t^3 - 20t$

    Distances are measured in ft.

22. A spherical bubble of volume 1 c.c. contains a gas at a pressure 5 cms. Its volume changes with the pressure given by the relation $PV^{1.4} = k$, where k is a real constant. If the pressure varies at the rate of ·02 cm/sec, find the rate of change of the volume when $V = 2^5$ ccs.

23. Find the point at which the tangent to the curve $y^2 = 4ax$ is parallel to the y axis

24. Find the point at which the tangent to each of the following curves is

    (i) parallel to x axis
    (ii) parallel to y axis :
    (1) $x^2 + y^2 = 16$

(2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

25. A particle moves along a straight line and its position after t secs is given by $S = 10 + 3t - 8t^2 + t^3$

Find the acceleration of the particle at the two times when the velocity is zero. Distances are measured in ft.

ANSWERS

1. (i) $\{x/3 \leqslant x \leqslant 5\}$
   (ii) $\{x/-6 < x \leqslant 0\}$
   (iii) $\{x/-5 < x < -1\}$
   (iv) $\{x/-2 \leqslant x < 4\}$
   (v) $\{x/a < x < y\}$
   (vi) $\{x/c < x \leqslant n\}$

2. (i) $(-1, 5)$, (ii) $(0, 8]$, (iii) $[6, 9]$.
   (iv) $(-2, 0]$, (v) $(-5, -1) \sim \{-3\}$
   (vi) $(2, 11)$, (vii) $(4, 13)$, (viii) $(-8, 3]$
   (ix) $[-1, 3]$, (x) $(-2\ 5)$, (xi) $(7, 9)$
   (xii) $[5, 6]$, (xiii) $\emptyset$, (xiv) ϕ (xv) $(-2, 0)$.
3. (i) true, (ii) true, (iii) false, (iv) true,
   (v) true, (vi) false, (vii) false.
4. Yes, Yes, No, Yes, No, No, Yes
5. C, C, A, A, B, A, C, B, B.
6. (i) 2, (ii) 4, (iii) 4, (iv) 2, (v) $2^m - 1$
   (vi) $\frac{1}{2}$, (vii) $\frac{11}{3}$, (viii) $-\frac{3}{2}$, (ix) $-\frac{4}{7}$.
   (x) $\frac{-3}{4\sqrt{2}\, a^{3/2}}$

7. (i) $-\sin\alpha$, (ii) 1, (iii) $\frac{m}{n}$, (iv) $\frac{\sqrt{2}}{4\pi^2}$

(v) $\frac{3}{2}$, (vi) $\frac{1}{2}$, (vii) $\sec^2 a$.

8. (a) 2, (b) $2x - 4$, (c) $5\cos 5x$, (d) $3\sec^2 3x$ (e) $\sec x \tan x$.

9. (i) $-13x^{-14}$, $25x^{24}$, $-\frac{9}{x^{10}}$

(ii) $\frac{-345}{x^{24}}$, $189x^6$, $\frac{10}{3x^3}$, (iii) $-3\sin x + 36x^3$.

(iv) $5\sec^2 x + \frac{15}{x^6} - \cos x$, (v) knx^{n-1}, (vi) $mc^m x^{m-1}$.

(vii) $10x^9 \sin x \cos x + x^{10}\cos^2 x - x^{10}\sin^2 x$

(viii) $15^{1/3} \cdot \frac{5}{3} x^{2/3} \tan x + \sqrt[3]{15x^5}\sec^2 x$

(ix) $85x^{16}\sec x + 5x^{17}\sec x \tan x$

(x) $3\left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right)$, (xi) $-[\operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x]$

(xii) $\frac{-[(ax+b)\sec^2 x + (3-\tan x)a]}{(ax+b)^2}$

11. (i) $4\left(x + \frac{1}{x}\right)^3\left(1 - \frac{1}{x^2}\right)$. (ii) $\frac{3}{2\sqrt{3x-5}}$.

(iii) $\frac{9}{(1-9x)^2}$, (iv) $\frac{-28}{(4x+15)^2}$, (v) $\frac{-3}{(x-1)^4}$

(vi) $3x^2 \cos x^3 + \frac{12x}{(3x^2+1)^2}$,

(vii) $30\tan^2 x \sec^2 x + 2x\sin x + x^2\cos x$.

(viii) $\frac{x^2(2x^2+15)}{(x^2+5)^{3/2}}$, (ix) $\cos(ax^3+bx^2+d)^4$,

(x) $\frac{(5x^3+9)\cdot 2\sec^2 x \tan x - 15x^2 \sec^2 x}{(5x^3+9)^2}$.

(xi) $7\,(px^2 + 2qx + t)^{5/2}\,(px + q)$.

(xii) $\dfrac{-3\cos 5x \cos 3x + 5\,(1 - \sin 3x)\sin 5x}{\cos^2 5x}$

(xiii) $\dfrac{7\sec^2 7x}{2\sqrt{\tan 7x}}$, (xiv) $35mx^{m-1}\cos x^m - \dfrac{p}{2}\,(a + bx)^{\frac{p-2}{2}}$

(xv) $\dfrac{-5\sin x\,(x^n + a) - 5nqx^{n-1}\cos x}{(x^n + a)^{q+1}}$

(xvi) $2\cos x \sin x\,(\cos^4 x - 2\cos^2 x \sin^2 x)$

(xvii) $\dfrac{2x\,(1 + x^4)}{(1 - x^4)^2}$, (xviii) $\dfrac{-23}{(x-9)^2}$

(xix) $\dfrac{3}{2}\,\dfrac{x^2\,(3x - 1)}{\sqrt{x^3 + 2}}$.

(xx) $10\,(x + a)^6\,[6\cos(bx + d)^3 - 3b\,(bx + d)^2 \sin(bx+d)^3]$

13. (i) $\dfrac{-3}{2}$, (ii) $\dfrac{-4}{3}$, (iii) -1, (iv) 4, (v) $\dfrac{1}{3}$.

14. 6π cm²/sec , 15π cm³/sec .

15. ·05 amps/sec

16. ·05 cm/sec .

17. 436 , 571

18. ·2 cm/sec

19. $4\frac{1}{2}$ cms/sec .

20. 25 m. p. h , 60 miles per hour per hour.

21. (a) vel : (i) 20 ft/sec , (ii) 50 ft/sec; Acc : (i) 10 ft/sec² , (ii) 10 ft/sec²

(b) vel : (i) 2 ft/sec , (ii) 8 ft/sec ,
Acc: (i) 2 ft/ sec² , (ii) 2 ft/sec² .

(c) vel: (i) 4 ft/sec ,
(ii) 130 ft/sec , Acc: (i) 24 ft/sec² , (ii) 60 ft/sec² .

22. 11·7 c. c. s/sec approximately .

23. $(0, 0)$

24. 1. (i) $(0, 4)$, $(0, -4)$, (ii) $(4, 0)$, $(-4, 0)$
2. (i) $(0, b)$, $(0, -b)$, (ii) $(a, 0)$, $(-a, 0)$

25. -8 ft/sec^2 , 8 ft/sec^2 .

CHAPTER EIGHT

Integration as Antidifferentiation

8·1. Consider the following examples :

1. $\frac{d}{dx} x^3 = 3x^2$
2. $\frac{d}{dx} \sin x = \cos x$
3. $\frac{d}{dx} \tan x = \sec^2 x$

Thus,

derivative of x^3 is $3x^2$
derivative of sin x is cos x
derivative of tan x is $\sec^2 x$.

We may think of these results the other way round as :

1. $3x^2$ is a function obtained by differentiating x^3 ;
2. cos x is a function obtained by differentiating sin x ;
3. $\sec^2 x$ is a function obtained by differentiating tan x.

Instead of expressing these results in all these words, we generally use the word "primitive" and write

1. $3x^2$ is a primitive of x^3.

2. $\cos x$ is a primitive of $\sin x$.
3. $\sec^2 x$ is a primitive of $\tan x$.

More often, we express the notion of primitive in terms of an operation such that starting with the primitive, applying the operation, we arrive at the resulting function whose derivative the primitive was. This operation is called the *operation of integration*. Thus, the examples stated above can be equivalently written as

1. Integration of $3x^2$ is x^3.
2. Integration of $\cos x$ is $\sin x$.
3. Integration of $\sec^2 x$ is $\tan x$.

For the sake of convenience, the operation of integration is denoted by the symbol $\int$.

$$\therefore \int 3x^2\,dx = x^3$$
$$\int \cos x\,dx = \sin x$$
$$\int \sec^2 x\,dx = \tan x$$

Note that $\cos x\,dx$ doesnot mean $\cos x$ multiplied by dx. In fact, dx is not a number at all. It is a symbol, and together with the symbol $\int$, it is used as a bracket, to denote the operation of integration. The function to be integrated is enclosed between $\int$ and dx.

Definition : If $\frac{d}{dx} f(x) = F(x)$, then $f(x)$ is said to be an integral of $F(x)$ and it is written as $\int F(x)\,dx = f(x)$

The function $F(x)$ which is integrated is called the integrand.

Integration thus defined, is an operation which is inverse of differentiation. It is, therefore, also called anti-differentiation.

That integration and differentiation produce exactly opposite effects on a function is apparent from the fact that if both are applied successively on a function the result is to get back the original function. Consider, for instance, $\int \frac{d}{dx} f(x)dx$ where $f(x)$ is any real valued function.

Let $\frac{d}{dx} f(x) = \phi(x)$ so that $\int \phi(x)\,dx = f(x)$

$$\therefore \int \frac{d}{dx} f(x)\,dx = \int \phi(x)\,dx$$

$$= f(x)$$

8·2. Constant of integration :

In the last chapter we have seen that

$$\frac{d}{dx} \tan x = \sec^2 x$$

and the derivative of tan x cannot have any other value. In fact, the derivative of any function of x has a unique value. This, however, is not true for the operation of integration.

Consider, $\frac{d}{dx} \sin x = \cos x \quad \therefore \int \cos x\,dx = \sin x$

$$\frac{d}{dx} (\sin x + 13) = \cos x \quad \therefore \int \cos x\,dx = \sin x + 13.$$

$$\frac{d}{dx} (\sin x + 205) = \cos x \quad \therefore \int \cos x\,dx = \sin x + 205$$

In general, $\frac{d}{dx} (\sin x + k) = \cos x$

$$\therefore \int \cos x\,dx = \sin x + k$$

where k is any constant real number.

Thus the integral of cos x is not unique. The operation of integration performed on cos x yields different results depending upon the function on which the inverse operation of differentiation is done. It should be noted that the different values of $\int \cos x\,dx$ differ only by some constant. As far as the function of x in the result is concerned, it is always sin x.

To take care of this aspect of antidifferentiation whenever it is given that

$$\frac{d}{dx} f(x) = F(x)$$

the inverse operation is always written as

$$\int F(x)\,dx = f(x) + k$$

where k is any constant. The actual value of this constant varies from context to context and depends on the particular conditions of the problem at hand. In the same context however, it remains the same throughout the discussion.

k is called the constant of integration. In problems where integration is done under specific particular conditions, the constant of integration plays a very important role. As far as the scope of our discussion of integration is concerned, however, we shall be interested more in the type of the function of x we obtain on integrating, than in the actual value of the constant of integration. To simplify matters therefore, we shall assume that we are always performing integrations under conditions which yield k = 0. So, unless the conditions of integration are specifically stated to be otherwise, the constant of integration will always be taken to be equal to zero and therefore, never written down. It should however, always be remembered that the constant of integration does exist in every case of integration.

8·3. Some Standard Integrals :

1. $$\int x^n\,dx = \frac{x^{n+1}}{n+1},\ n \neq -1,\ n \in I$$

Since $\frac{d}{dx} x^{n+1} = (n+1)\,x^n$

$$\therefore\ \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1}\,\frac{d}{dx}\,x^{n+1}$$

$$= \frac{1}{n+1}\,(n+1)\,x^n$$

$$= x^n$$

$$\therefore\ \int x^n dx = \frac{x^{n+1}}{n+1},\ n \neq -1.$$

2. $$\int \cos x\ \ dx = \sin x.$$

3. $\int \sin x \, dx = -\cos x.$

4 $\int \sec^2 x \, dx = \tan x$

5. $\int \operatorname{cosec}^2 x \, dx = -\cot x$

6. $\int \sec x \tan x \, dx = \sec x$

7 $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x.$

These results can be obtained directly from the standard results of derivatives.

Examples :

Integrate (i) x^3 (ii) $\frac{1}{x^9}$ (iii) $\frac{\cos x}{\sin^2 x}$ with respect to x.

(i) $\int x^3 \, dx = \frac{x^{3+1}}{3+1} = \frac{x^4}{4}$

(ii) $\int \frac{1}{x^9} \, dx = \int x^{-9} dx = \frac{x^{-9+1}}{-9+1} = \frac{x^{-8}}{-8}$

$$= -\frac{1}{8} \cdot x^{-8}$$

(iii) $\int \frac{\cos x}{\sin^2 x} \, dx = \int \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} \, dx$

$$= \int \cot x \cdot \operatorname{cosec} x \, dx$$

$$= -\operatorname{cosec} x.$$

2. If $\phi(x)$ and $\psi(x)$ are any two real valued functions of x such that $\phi^1(x) = \psi^1(x)$ then prove that $\int \phi^1(x) \, dx = \int \psi^1(x) \, dx + k$. where k is a real constant.

Let $\phi^1(x) = \psi^1(x) = f(x)$

Also let $\int f(x)\,dx = F(x) +$ a constant,

$$\therefore \int \phi^1(x)\,dx = \int f(x)\,dx = F(x) + c_1$$

and $\int \psi^1(x)\,dx = \int f(x)\,dx = F(x) + c_2$ where c_1 and c_2 are real constants.

$$\therefore \int \phi^1(x)\,dx - \int \psi^1(x)\,dx = c_1 - c_2$$

$$\therefore \int \phi^1(x)\,dx = \int \psi^1(x)\,dx + k.$$

8·4. Rules for integration :

Rule 1. The integral of the product of a constant and a function of x is equal to the product of the constant and the integral of the function of x.

Let f(x) be any function of x. We wish to show that for any real constant k, $\int k\,f(x)\,dx = k\int f(x)\,dx$.

$$\text{Let } \int f(x)\,dx = F(x) \quad \ldots \quad \text{(i)}$$

$\therefore$ By definition,

$$\frac{d}{dx}F(x) = f(x) \quad \ldots \quad \text{(ii)}$$

$$\text{Now, } \frac{d}{dx}k\,F(x) = k\frac{d}{dx}F(x)$$

$$= k\,f(x) \text{ from (ii)}$$

$$\therefore \int k\,f(x)\,dx = k\,F(x) \text{ By Definition.}$$

$$= k\int f(x)\,dx \text{ from (i)}$$

Illustration :

$$\int 5x^{12}\,dx = 5\int x^{12}\,dx$$

$$= 5\frac{x^{12+1}}{12+1}$$

$$= \frac{5}{13}x^{13}.$$

Rule 2 : The integral of the sum of two or more functions of x is the same as the sum of the integrals of the individual functions.

We shall prove the rule for any two real valued functions $f_1(x)$ and $f_2(x)$ of x.

We wish to show that

$$\int\{f_1(x) + f_2(x)\}\,dx = \int f_1(x)\,dx + \int f_2(x)\,dx.$$

Let $\int f_1(x)\,dx = \phi(x)$.

and $\int f_2(x)\,dx = \psi(x)$

$\therefore$ By definition, $\frac{d}{dx}\phi(x) = f_1(x)$

and $\frac{d}{dx}\psi(x) = f_2(x)$.

$$\text{Now, } \frac{d}{dx}\{\phi(x) + \psi(x)\} = \frac{d}{dx}\phi(x) + \frac{d}{dx}\psi(x)$$

$$= f_1(x) + f_2(x).$$

$$\therefore \int\{f_1(x) + f_2(x)\}\,dx = \phi(x) + \psi(x)$$

$$= \int f_1(x)\,dx + \int f_2(x)\,dx.$$

This rule can be easily extended, so that for a finite number of functions,

$$\int\{f_1(x) + f_2(x) + \ldots\ldots f_n(x)\}dx$$

$$= \int f_1(x)\,dx + \int f_2(x)\,dx + \ldots\ldots\ldots + \int f_n(x)\,dx$$

Note: Obviously, $\int\{f_1(x) - f_2(x)\}\,dx = \int f_1(x)dx - \int f_2(x)\,dx$.

for, $\int\{f_1(x) - f_2(x)\}\,dx = \int[f_1(x) + \{-f_2(x)\}]\,dx$

$$= \int f_1(x)\,dx + \int(-1)\,f_2(x)\,dx$$

$$= \int f_1(x)\,dx + (-1)\int f_2(x)\,dx$$

$$= \int f_1(x)\,dx - \int f_2(x)\,dx.$$

Illustrations :

1. Integrate $\frac{(x + a)^2}{x^7}$ with respect to x.

$$\int \frac{(x+a)^2}{x^7}\,dx = \int \frac{x^2+2x\,a+a^2}{x^7}\,dx$$

$$= \int \{x^{-5} + 2x^{-6}a + x^{-7}a^2\}\,dx$$

$$= \int \{x^{-5}dx + \int 2x^{-6}a\,dx + \int x^{-7}a^2\,dx$$

$$= \int x^{-5}dx + 2a\int x^{-6}dx + a^2\int x^{-7}dx$$

$$= \frac{x^{-4}}{-4} + 2a\,\frac{x^{-5}}{-5} + a^2\,\frac{x^{-6}}{-6}$$

$$= -\frac{x^{-4}}{4} - \frac{2}{5}\,a\,x^{-5} - \frac{a^2}{6}\,x^{-6}$$

2. Integrate $\int \tan^2\theta\ d\theta$

$$\int \tan^2\theta\ d\theta = \int (\sec^2\theta - 1)\,d\theta$$

$$= \int \sec^2\theta\ d\theta - \int 1\cdot d\theta$$

$$= \tan\theta - \theta.$$

[Note; $\int dx = \int \mathbf{1}\,dx = \int x^0\,dx = \frac{x^{0+1}}{0+1} = x$].

3. If $f^1(x) = 5x^2 + \frac{1}{x^3}$, find the equation to the curve $y = f(x)$ which passes through the point (1, 2).

Since $f^1(x) = 5x^2 + \frac{1}{x^3}$

$$\therefore\ f(x) = \int\left(5x^2 + \frac{1}{x^3}\right)dx.$$

$$= 5\int x^2\,dx + \int x^{-3}\,dx$$

$$= 5\,\frac{x^3}{3} + \frac{x^{-2}}{-2} + c$$

where c is the constant of integration.

$\therefore\ y = f(x)$ is given by

$$y = 5\,\frac{x^3}{3} - \frac{1}{2x^2} + c \ \ldots\ldots. (1)$$

Since this passes through the point (1, 2)

$$\therefore \quad 2 = \frac{5}{3} - \frac{1}{2} + c.$$

$$\therefore \quad c = \frac{5}{6}$$

$\therefore$ the equation (i) becomes

$$y = 5\frac{x^3}{3} - \frac{1}{2x^2} + \frac{5}{6}$$

i. e. $6x^2y = 10x^5 - 3 + 5x^2$

i. e. $10x^5 - 6x^2y + 5x^2 = 3.$

Note that the constant of integration is to be determined since a particular condition for the function $y = f(x)$ is specifically stated.

8·5. Techniques of integration:

One of the powerful and frequently used methods for solving integrals is that of substitution. When the given integration cannot directly be solved, the integrand, in whole or part is substituted as a function of a new variable. The whole integrand is then converted explicitly in terms of this new variable. Consider for instance, the integral $\int f(x)\,dx$, where $f(x)$ is a real valued function of x.

Substitute $x = \phi(t)$ and let $f(x)$, by this substitution, become $F(t)$.

Now, our aim is to find $y = \int f(x)\,dx$.

By definition, $\dfrac{dy}{dx} = f(x)$

$$\text{But} \quad \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\therefore \quad \frac{dy}{dt} \cdot \frac{dt}{dx} = f(x) = F(t)$$

$$\therefore \quad \frac{dy}{dt} = F(t)\,\frac{dx}{dt}$$

$$\therefore \quad \text{By definition, } y = \int F(t)\,\frac{dx}{dt} \cdot dt.$$

$$\text{So, } \int f(x)\,dx = \int F(t)\cdot\frac{dx}{dt}\cdot dt.$$

The types of substitution are varied and many, of which the following are some.

Type 1. Integrals of the form $\int (ax+b)^n\,dx$. $n \in R$, $n \neq -1$.

Let $ax + b = t$, where t is a variable depending on x.

Differentiating both sides w.r.t. t,

$$a\cdot\frac{dx}{dt} = 1 \quad \therefore \frac{dx}{dt} = \frac{1}{a} \quad \ldots\ldots.(1)$$

$$\text{Now, } \int (ax+b)^n\,dx = \int (ax+b)^n\,\frac{dx}{dt}\cdot dt.$$

$$= \int t^n.\ \frac{1}{a}\,dt \cdot \text{ from (i)}$$

$$= \frac{1}{a}\int t^n\,dt.$$

$$= \frac{1}{a}\cdot\frac{t^{n+1}}{n+1} = \frac{(ax+b)^{n+1}}{a(n+1)}.$$

In general, to integrate a function of any linear function of x, we always substitute the given linear function by a new variable and proceed as above.

Illustrations : Integrate with respect to x,

(i) $\int \sin mx\,dx$ (ii) $\int \sqrt{(2x+7)}\,dx$

(iii) $\int \frac{1}{\sqrt{x} - \sqrt{x-1}}$

(i) $\int \sin mx\,dx$

$$\text{put } mx = t. \quad \therefore m\,\frac{dx}{dt} = 1 \quad \therefore \frac{dx}{dt} = \frac{1}{m}$$

$$\int \sin\ mx\ dx = \int \sin\ mx\ \frac{dx}{dt} \cdot dt.$$

$$= \int \sin t \cdot \frac{1}{m} \cdot dt.$$

$$= \frac{1}{m} \int \sin t\ \ dt.$$

$$= \frac{1}{m} (-\cos t)$$

$$= -\frac{1}{m} \cos mx,$$

(ii) $\int \sqrt{2x + 7)}\ dx.$

put $2x + 7 = t \quad \therefore\ 2 \frac{dx}{dt} = 1 \quad \therefore\ \frac{dx}{dt} = \frac{1}{2}$ ∴

$$\therefore\ \int \sqrt{(2x+7)}\ dx = \int (2x + 7)^{1/2} \frac{dx}{dt} . dt$$

$$= \int t^{1/2}. \frac{1}{2} \cdot dt$$

$$= \frac{1}{2} \cdot \frac{t^{1/2+1}}{\frac{1}{2}+1}$$

$$= \frac{1}{2} \cdot \frac{t^{3/2}}{\frac{3}{2}}$$

$$= \frac{1}{3} (2x + 7)^{3/2}$$

(iii) $\int \frac{1}{\sqrt{x} - \sqrt{x-1}}\ dx = \int \frac{1}{\sqrt{x} - \sqrt{x-1}} \times \frac{\sqrt{x} + \sqrt{x-1}}{\sqrt{x} + \sqrt{x-1}}\ dx$

$$= \int \frac{\sqrt{x} + \sqrt{x-1}}{x - (x - 1)}\ dx$$

$$= \int (\sqrt{x} + \sqrt{x-1})\,dx$$

$$= \int x^{1/2}\,dx + \int (x-1)^{1/2}\,dx$$

$$= \frac{x^{3/2}}{\frac{3}{2}} + \frac{(x-1)^{3/2}}{\frac{3}{2}} \quad \text{[use Art. 8·5]}$$

$$= \frac{2}{3}\,[\,x^{3/2} + (x-1)^{3/2}]$$

(iv) If $\frac{ds}{dt} = (t-1)^2$ and $s=2$ when $t=1$, express s in terms of t.

Integrating with respect to t,

$$\int \frac{ds}{dt}\,dt = \int (t-1)^2\,dt$$

$$\therefore \int \left(\frac{d}{dt}\,s\right) dt = \int (t^2 - 2t + 1)\,dt$$

$$\therefore s = \frac{t^3}{3} - 2\cdot\frac{t^2}{2} + t + k$$ Where k is the constant of integration.

Since $s = 2$ when $t = 1$,

$$\therefore 2 = \frac{1}{3} - 1 + 1 + k \quad \therefore k = 2 - \frac{1}{3} = \frac{5}{3}$$

$$\therefore s = \frac{t^3}{3} - t^2 + t + \frac{5}{3}\cdot$$

$$\therefore 3s = t^3 - 3t^2 + 3t + 5$$

Type 2 : Integrals containing the product of a trigonometric function of x and its derivative.

In such cases, always substitute the trigonometric function by a new variable.

e.g. To integrate $\tan^p x \sec^2 x$ with respect to x, put $\tan x = t$.

so that, $\sec^2 x \dfrac{dx}{dt} = 1.$

$$\int \tan^p x \sec^2 x \, dx = \int \tan^p x \text{ see}^2 x \dfrac{dx}{dt} \cdot dt$$

$$= \int t^p \cdot 1 \, dt$$

$$= \frac{t^{p+1}}{p+1}$$

$$= \frac{\tan^{p+1} x}{p+1}.$$

8·6. More trigonometric types.

1. $$\int \cos^2 x \, dx = \int \frac{1 + \cos 2x}{2} \, dx.$$

$$= \frac{1}{2} \left[\int dx + \int \cos 2x \, dx \right]$$

$$= \frac{1}{2} \left[x + \frac{\sin 2x}{2} \right] \text{ (By putting } 2x = t \text{ etc.)}$$

$$= \frac{1}{2} x + \frac{1}{4} \sin 2x$$

2. To integrate the product of any two linear trigonometric functions of x, express it as a sum or difference of trigonometric functions.

e. g., $$\int \sin 7x \cos 3x \, dx$$

$$= \frac{1}{2} \int 2 \sin 7x \cos 3x \, dx$$

$$= \frac{1}{2} \int \{ \sin (7x + 3x) + \sin (7x - 3x) \} \, dx$$

$$= \frac{1}{2} \left[\int \sin 10x \, dx + \int \sin 4x \, dx \right]$$

$$= \frac{1}{2}\left[\frac{-\cos 10x}{10} - \frac{\cos 4x}{4}\right]$$

$$= -\frac{1}{20}\cos 10x - \frac{1}{8}\cos 4x.$$

8·7. Definite integral :

Definition : If $\int f(x)\,dx = F(x)$ then $\int_a^b f(x)\,dx$ is the symbol that stands for F(b) – F(a) where a and b are any two real constants.

$$\text{So, } \int_a^b f(x)\,dx = F(b) - F(a)$$

The integral $\int_a^b f(x)\,dx$ is called a definite integral of f(x) from a to b. The numbers a and b are called the limits of f(x); a is called the lower limit and b is called the upper limit.

To find the definite integral of a function between two given limits, the ordinary integral of the function is first to be found. The difference of the values of the integrated function at the upper and lower limits is the result of the definite integral of the function between the two given limits.

Note that the constant of integration does not play any role for a definite integral under any condition.

For, let $\int f(x)\,dx = F(x) + c$, where c is the constant of integration .

$$\therefore \int_a^b f(x)\,dx = \Big[F(x) + c\Big]_{at\ x=b} - \Big[F(x) + c\Big]_{at\ x=a}$$

$$= \Big[F(b) + c\Big] - \Big[F(a) + c\Big]$$

$$= F(b) - F(a)$$

the constant c remaining the same at $x = b$ and $x = a$.

Illustrations :

Evaluate the following definite integrals:

(1) $\displaystyle\int_1^2 \left(x^5 + 3x^3 - \frac{1}{x^2}\right) dx$

$$= \left[\frac{x^6}{6} + \frac{3x^4}{4} - \frac{x^{-1}}{(-1)}\right]_1^2$$

$$= \left(\frac{2^6}{6} + \frac{3}{4}\cdot 2^4 + \frac{1}{2}\right) - \left(\frac{1}{6} + \frac{3}{4} + 1\right)$$

$$= \frac{85}{4}\ .$$

(2) Evaluate $\displaystyle\int_0^{\pi/2} \cos^3 x\ dx$

$$\int_0^{\pi/2} \cos^3 x\, dx = \int_0^{\pi/2} \cos^2 x \cdot \cos x\ dx$$

$$= \int_0^{\pi/2} (1 - \sin^2 x) \cos x\ dx$$

$$= \int_0^{\pi/2} \cos x \, dx - \int_0^{\pi/2} \sin^2 x \cos x \, dx$$

$$= \Big[\sin x \Big]_0^{\pi/2} - \int_0^{\pi/2} \sin^2 x \cos x \, dx \quad \ldots \ (1)$$

Now, $\Big[\sin x \Big]_0^{\pi/2} = \sin \frac{\pi}{2} - \sin 0 = 1$

For the other integral i. e, $\int_0^{\pi/2} \sin^2 x \cos x \, dx$

put $\sin x = t$

$\therefore \cos x \frac{dx}{dt} = 1$

when $x = 0$, $t = 0$ and when $x = \pi/2$, $t = 1$.

$\therefore$ From (1) $\int_0^{\pi/2} \cos^3 x \, dx = 1 - \int_0^1 t^2 \, dt$

$$= 1 - \Big[\frac{t^3}{3} \Big]_0^1$$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3} .$$

(3) $\int_0^a 3x^2 \sin x^3 \, dx$

put $x^3 = t$

$\therefore \; 3x^2 \frac{dx}{dt} = 1$

$\therefore \; \frac{dx}{dt} = \frac{1}{3x^2}$

At $x = 0$, $t = 0$ and at $x = a$, $t = a^3$

$$\int_0^a 3x^2 \sin x^3 \, dx = \int_0^{a^3} \sin t \, dt$$

$$= \Big[-\cos t \Big]_0^{a^3}$$

$$= -\Big[\cos a^3 - \cos 0\Big]$$

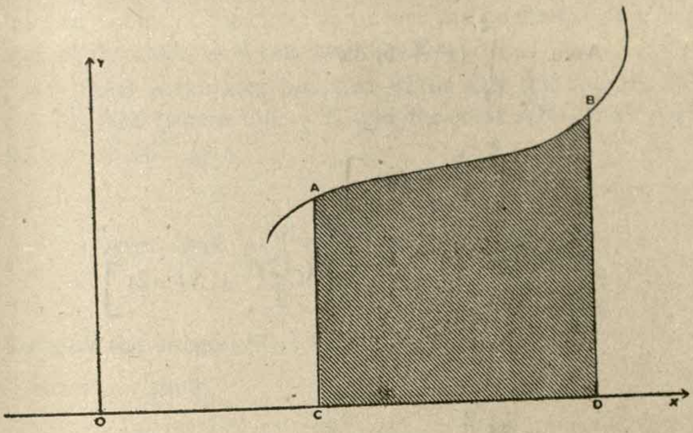
$$= 1 - \cos a^3 .$$

8·8. Applications of integration :

The applications of integration are varied and many. Of the numerous applications of integration we give below only two, through illustrative examples.

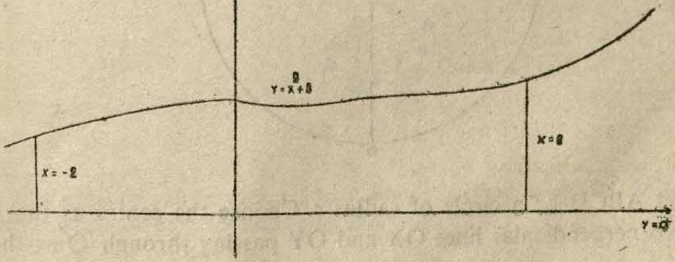
A definite integral, though we shall not prove it, is the limit of a summation. In fact, the notion of integration was first discovered and formalised in an attempt to find areas of circles etc. Integration as an inverse process of differentiation was established much later.

1. **Area beneath a curve :** Let AB represent the graph $y = f(x)$, OC = a, OD = b and CA and BD be the lines through C and D parallel to y axis. We define the area bounded by AC, BD, x axis and the curve



(that is, the shaded portion) to, be $\int_{a}^{b} f(x)\,dx$. Similarly, the area enclosed between the y axis, a curve x = f(y) and the lines x = c, x = d is $\int_{c}^{d} f(y)\,dy$.

Example : Find the area between $y = 0$, $y = x^2 + 5$, $x = -2$ and $x = 3$ ·



$$\text{Area} = \int_{-2}^{3} (x^2 + 5)\,dx$$

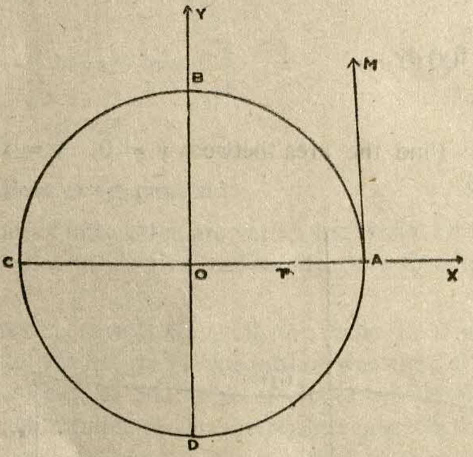
$$= \left[\frac{x^3}{3} + 5x \right]_{-2}^{3}$$

$$= \frac{3^3}{3} + 5\cdot3 - \left[\frac{(-2)^3}{3} + 5(-2) \right]$$

$$= \frac{110}{3}$$

$$= 36\,\frac{2}{3}$$

2. Area of a circle : To find the area bounded within the circumference of a circle.



Let ABCD be a circle of radius r. Choose the centre as the origin and two perpendicular lines OX and OY passing through O as the axes

of reference. The equation to the circle is $x^2 + y^2 = r^2$*. r remains the same at every point of the circle and hence can be treated as a constant. The area of the circle = 4 (the area OAB)....(i)
Now, area OAB is the area bounded within OA (the x-axis), OB (the line $x = 0$), AM (where $OA = r$) and the curve AB (i.e. $x^2 + y^2 = r^2$, that is, $y = \sqrt{r^2 - x^2}$)

$$\therefore \quad \text{Area OAB} = \int_0^r \sqrt{r^2 - x^2}\, dx \ldots . \text{(ii)}$$

To solve the integral (ii),

put $x = r \sin \theta$

$$\therefore \quad \frac{dx}{d\theta} = r \cos \theta \text{ (remember r is a constant)}$$

at the lower limit, $x = 0 \quad \therefore \theta = 0.$

at the upper limit, $x = r \quad \therefore \theta = \frac{\pi}{2}$ (since the area OAB is confined to the first quadrant)

$$\therefore \text{ from (ii), Area OAB} = \int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2 \sin \theta} \cdot r \cos\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} r^2 \cos^2 \theta \, d\theta$$

$$= r^2 \cdot \frac{1}{2} \cdot \int_0^{\frac{\pi}{2}} 2 \cos^2 \theta d\theta$$

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* This equation can be obtained by a simple method and may be found in any elementary book on plane coordinate geometry.

$$= \frac{r^2}{2} \int_0^{\frac{\pi}{2}} (\cos 2\theta + 1)\, d\theta$$

$$= \frac{r^2}{2} \left[\left(\theta + \frac{\sin 2\theta}{2} \right) \right]_0^{\pi/2}$$

$$= \frac{\pi r^2}{4}$$

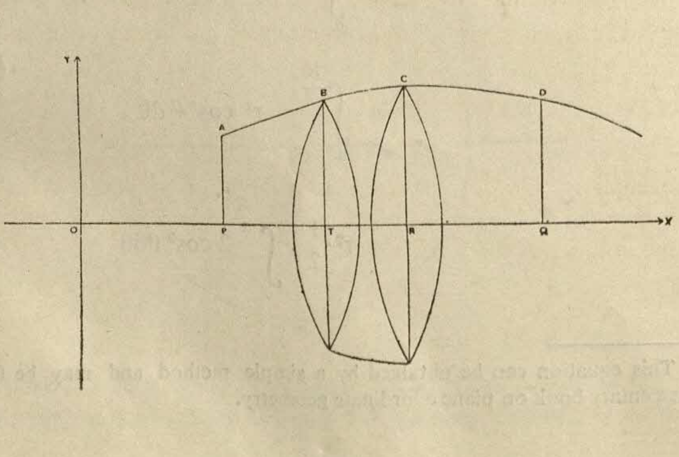
∴ from (i),

$$\text{Area of the circle} = 4 \left(\frac{\pi r^2}{4} \right)$$

$$= \pi r^2$$

3. Solids of revolution :

Let the curve ABCD represent the graph of $y = f(x)$ and the line PA and QD represent the lines $x = a$ and $x = b$ respectively. Suppose the area PABCDQ is revolved through four right angles about the x-axis Consider two points B and C very near each other on the given curve and let BT and CR be the perpendiculars drawn from B and C resep



tively on the axis of x. As the area PADQ revolves about the x-axis, each point on the curve AD generates a circle about a corresponding point on the x-axis. The points B and C generate a circle each about T and R as respective centres.

Let TR = $\triangle x$, BT = y and CR = $y + \triangle y$. Then the area of the circle generated by B $= \pi\,(BT)^2$

$= \pi\, y^2$.

and the area of the circle generated by C $= \pi\,(CR)^2 = \pi\,(y + \triangle y)^2$.

$\therefore$ If $\triangle V$ be the solid formed by revolution of the arc BC about x-axis, then

$$\pi\, y^2 \triangle x < \triangle V < \pi\,(y + \triangle y)^2 \triangle x$$

The volume of the solid generated by revolution of the arc AD about x-axis is the sum of all such volumes like $\triangle V$, and therefore, if V be this volume, then sum of all volumes like $\pi y^2 \triangle x < V <$ sum of all volumes like $\pi(y + \triangle y)^2 \triangle x$. Hence the volume formed by revolving the area bounded between the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$ about the axis of x is **defined** as

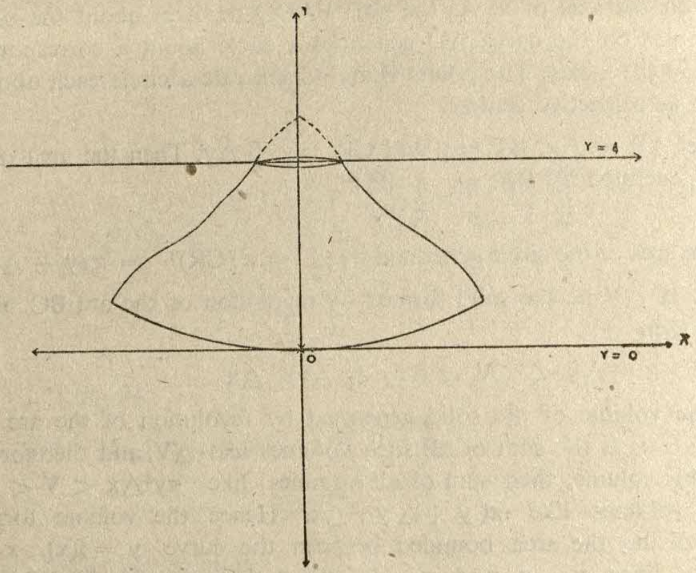
$$V = \int_a^b \pi\, y^2\, dx\,.$$

Similarly, the volume of a solid of revolution formed by a complete revolution of the curve $x = f(y)$ bounded by the lines $y = c$ and $y = d$

about OY is $\int_c^d \pi\, x^2\, dy$.

Example : A vase is formed by rotating about the y axis the part of the curve $x^2 = y\,(y - 5)^2$ that lies between $y = 0$ and $y = 4$. Find the volume of the vase.

$$V = \int_0^4 \pi \cdot y \cdot (y - 5)^2\, dy\,.$$



$$= \pi \int_{0}^{4} (y^3 - 10y^2 + 25\,y)\,dy$$

$$= \pi \left[\frac{y^4}{4} - 10\frac{y^3}{3} + 25\frac{y^2}{2} \right]_0^4$$

$$= 152\frac{\pi}{3}.$$

EXERCISE

Integrate the following with respect to x,

1. $5x^7, x^{-3}, \frac{1}{2}x^{3/2}, \frac{1}{\sqrt{x^5}}, x^{-7/2}$

2. $6x^2 + 9,\ x^4 - 20x + 13,\ ax^2 + b,\ \frac{c}{x^9} + d$

3. $\frac{(x-1)^2}{x^4},\ \sqrt{x}\ (x^3+3)^2,\ \frac{x}{(x^3-2)^{-2}},\ \frac{(2x+1)(x-4)}{\sqrt{x^3}}$

4. $\left(\sqrt{x} + \frac{1}{x^3}\right)\left(x - \frac{3}{\sqrt{x}}\right)$

5. If $\phi^1(x) = 2x^3 + x - 1$, find $\phi(x)$ with the condition that $\phi(0) = 1$

Integrate with respect to x :

6. $(x+5)^4,\ (5x+9)^{10},\ \frac{1}{(a+bx)^2},\ \sqrt{(2+x)^3},\ \sqrt{3-2x},$

$\frac{1}{(3-4x)^3}$

7. $\frac{1}{\sqrt{2x+3} + \sqrt{2x-7}}$ [Hint : rationalise]

8. (i) $\frac{3x+9}{\sqrt{3x+1}}$ $\left[\text{Hint : } \frac{3x+9}{\sqrt{3x+1}} = \frac{3x+1}{\sqrt{3x+1}} + \frac{8}{\sqrt{3x+1}}\right]$

(ii) $\frac{x}{\sqrt{2-5x}}$ (iii) $\frac{x+5}{\sqrt{x+2}}$.

9. If $F^1(x) = cx^2 + \frac{x}{3} - 1$, and $F(0) = 2$ and

$F(1) = \frac{1}{6}$, find F(x).

[Hint : Take into consideration the constant of integration].

10. The velocity of a particle moving along a straight line is given by $v = 4t^2 + 3t - 1$. Find an expression for S, the distance travelled by the particle, in terms of t, the time taken to traverse the distance S, given that the distance is measured in feet, from the point the particle was at $t = 0$ (i.e. $S = 0$ at $t = 0$). Also find S when $t = 4$ seconds.

11. Integrate (i) $\int\left(3-\frac{1}{x}\right)\cdot\frac{1}{x^2}\,dx$ (ii) $\int(2x+3)(x^2+3x)^5dx.$

Integrate the following.

12. (i) $\int \tan^3 z \sec^2 z\,dz$ (ii) $\int \sin^4 x \cos x\,dx.$

(iii) $\int(6-7\cot u)^3 \operatorname{cosec}^2 u\,du.$ (iv) $\int \cos^2 x \sin x\,dx.$

(v) $\int \frac{\sec^2 v}{(2+\tan v)^3}\,dv.$

13. $\int \sin^2 x\,dx$

14. $\int \sin^3 x\,dx$

15. $\int \frac{\sin x}{\sqrt{4-5\cos x}}\,dx$

16. $\int \tan x \sec^6 x\,dx$ [Hint : $\tan x \sec^6 x = (\tan x \sec x) \sec e^5$ put $\sec x = t$]

17. $\int \frac{\sin^2 x}{\cos^4 x}\,dx$ [Convert in tan x and sec x]

18. $\int 3\cos 5x \sin 7x\,dx$

19. $\int \frac{1}{17} \cos 2x \cos 3x\,dx.$

20. $\int \sin 9x \sin 11x\,dx$

Integrate the following by the substitutions suggested.

21. $\int \sqrt{\frac{x}{1-x}}\,dx \left[\text{Substitute } x = \text{Sin}^2 u,\ 0 < u < \frac{\pi}{2}\right]$

22. $\int \frac{du}{1+u^2}$ $\left[\text{Substitute } u = \tan y, \quad 0 < y < \frac{\pi}{2}\right]$

23. $\int \frac{dz}{1-z^2}$ $\left[\text{Substitute } z = \sin v, \quad 0 < v < \frac{\pi}{2}\right]$

24. $\int x\sqrt{1-x}\ dx$ [Substitute $1-x = t^2$, $t > 0$]

25. Integrate $\int x\sqrt{x+1}\ dx$ [Hint : put $x = u^2 - 1$]

26. Integrate $\int \sqrt{1-x^2}\ dx$

27. Integrate $\int (1 + x^2)^{-3/2}\ dx$

28. A particle, starting from rest, moves with an acceleration $4t-14$. Express its velocity at any instant in terms of the time t, and find after what time it will come to rest again.

29. Evaluate the following integrals.

(i) $\int_0^3 x^{-1/2}\ dx$ (ii) $\int_{-1}^1 (ax^3 + bx^2 + cx + d)\ dx$

(iii) $\int_2^a (5x^7 - 2x^4)\ dx$ (iv) $\int_0^2 \frac{dx}{\sqrt{(1+7x)^3}}$ (v) $\int_0^c (x^2-c^2)dx$

30. Prove that

$$\int_a^b f(x)\ dx = \int_a^b f(t)\ dt.$$

31. Prove that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

32. Prove that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

33. Evaluate

(i) $\int_0^{\pi/4} \sec^2 x\,dx$. (ii) $\int_0^{\pi/2} (\sin 5\theta - 4\sin^2\theta)\,d\theta$

(iii) $\int_0^{\pi/2} \sin\theta\,\cos^3\theta\,d\theta$ (iv) $\int_0^{\pi/8} \sin\frac{5x}{2}\,\sin\tfrac{3}{2}x\,dx$

(v) $\int_0^{\pi/4} \sin^2 2x\,\cos 2x\,dx$

34. Show that $\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{4}$

35. If $\int_0^m (2x-5)\,dx = -6$, Find m.

36. If $\int_b^0 (y-2)^2\,dy = -\frac{35}{3}$, Find b.

37. The velocity of a particle which starts from rest at a point 0 and moves along a straight line passing through 0 is given by $v = 3t - t^2$, t being the time elapsed from the start. Show that the particle is at rest again after 3 seconds and find its distance from 0 at that time. Distances to be measured in cms.

38. Find the equation to the curve which passes through the origin and is such that at the point (x, y), $\frac{dy}{dx} = 7 - 6x + 5x^2$.

39. Solve the following equations, using the given conditions to determine the constant of integration.

(i) $\frac{dy}{dx} = 5\cos x + 24x^2$; at $x = 0$, y equals -3.

(ii) $x\,\frac{dy}{dx} = 7x^3 - 13x^2 + 9x$; $x = 1$ at $y = 0$.

40. A curve passes through the point $(-4, 9)$ and its slope at a point (x, y) is given by $\frac{dy}{dx} = 2x - 2$.

Find (i) the ordinate at $x = 4$ and (ii) the points where it crosses the x-axis, (iii), the point where it intersects the y-axis.

ANSWERS

1. $\frac{5}{8}x^8$, $-\frac{1}{2x^2}$, $\frac{1}{5}x^{5/2}$, $-\frac{2}{3}x^{-3/2}$, $-\frac{2}{5}x^{-5/2}$.

2. $2x^3 + 9x$, $\frac{1}{5}x^5 - 10x^2 + 13x$, $\frac{a}{3}x^3 + bx$, $\frac{-c}{8x^3} + dx$.

3. $\frac{-1}{x} + \frac{1}{x^2} - \frac{1}{3x^3}$, $\frac{2}{15}x^{15/2} + \frac{4}{3}x^{9/2} + 6x^{3/2}$,

$\frac{1}{8}x^8 - \frac{4}{5}x^5 + 2x^2$, $\frac{4}{3}x^{3/2} - 14x^{1/2} + 8x^{-1/2}$.

4. $\frac{2}{5}x^{5/2} - \frac{1}{x} - 3x + \frac{6}{5x^{5/2}}$

5. $\frac{1}{2}x^4 + \frac{1}{2}x^2 - x + 1$.

6. $\frac{1}{5}(x+5)^5$, $\frac{1}{55}(5x+9)^{11}$, $\frac{-1}{b(a+bx)}$,

$\frac{2}{5}\cdot(2+x)^{5/2}$, $-\frac{1}{3}(3-2x)^{3/2}$, $\frac{1}{8(3-4x)^2}$

7. $\frac{1}{30}[2x+3)^{3/2} - (2x-7)^{3/2}]$

8. (i) $\frac{2}{9}[(3x+1)^{3/2} + 24(3x+1)^{1/2}]$

(ii) $\frac{1}{25}\left[\frac{2}{3}(2-5x)^{3/2} - 4(2-5x)^{1/2}\right]$

(iii) $\frac{2}{3}(x+2)^{3/2} + 6(x+2)^{1/2}$

9. $-x^3 + \frac{1}{6}x^2 - x + 2$

10. $S = \frac{4}{3}t^3 + \frac{3}{2}t^2 - t$; $105\frac{1}{3}$ ft.

11. (i) $\frac{2}{3}\left(3 - \frac{1}{x}\right)^{3/2}$ (ii) $\frac{1}{6}(x^2 + 3x)^6$.

12. (i) $\frac{1}{4}\tan^4 z$ (ii) $\frac{1}{5}\sin^5 x$ (iii) $\frac{1}{28}(6 - 7\cot u)^4$

(iv) $-\frac{1}{3}\cos^3 x$ (v) $-\frac{1}{2(2+\tan v)^2}$

13. $\frac{1}{2}x - \frac{1}{4}\sin 2x$

14. $\frac{1}{3}\cos^3 x - \cos x$.

15. $\frac{2}{5}\sqrt{4-5\cos x}$

16. $\frac{1}{6}\sec^6 x$

17. $\frac{1}{3}\tan^3 x$

18. $-\frac{3}{24}(\cos 12x + 6\cos 2x)$

19. $\frac{1}{170}(\sin 5x + 5\sin x)$

20. $\frac{1}{40}(10\sin 2x - \sin 20x)$

12. $\sin^{-1}\sqrt{x} - \sqrt{x}\;\sqrt{1-x}$

22. $\tan^{-1} u$

32. $\sin^{-1} z$

24. $\frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2}$

25. $\frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2}$

26. $\frac{1}{2}\sin^{-1} x + \frac{1}{2}x\sqrt{1-x^2}$

27. $\frac{x}{\sqrt{1+x^2}}$

28. $v = 2t^2 - 14t$, 7 Seconds

29. (i) $2\sqrt{3}$, (ii) $\frac{2}{3}(b+3d)$, (iii) $\frac{5}{8}a^8 - \frac{2}{5}a^5 - \frac{736}{5}$

(iv) $\frac{2}{7}\left[1 - \frac{1}{\sqrt{15}}\right]$, (v) $-\frac{2}{3}c^3$

33. (i) 1, (ii) $\frac{1}{5} - \pi - \frac{1}{5} \cos^5 \frac{u}{2}$, (iii) $\frac{1}{4}$, (iv) $\frac{1}{2} \cos \frac{\pi}{8}$
(v) $\frac{1}{6}$.

35. 2, 3

36. $b = 5$

37. $4\frac{1}{2}$ cms.

38. $5x^3 - 9x^2 + 21x - 3y = 0$

39. (i) $y = 5 \sin x + 8 x^3 - 3$
(ii) $14 x^3 - 39 x^2 + 54 x - 6y = 29$

40. (i) -7, (ii) $(-3, 0)$ and $(5, 0)$,
(iii) $(0, -15)$.